

# Constacyclic codes of length $p^s n$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

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## Abstract

Let  $\mathbb{F}_{p^m}$  be a finite field of cardinality  $p^m$  and  $R = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  ( $u^2 = 0$ ), where  $p$  is a prime and  $m$  is a positive integer. For any  $\lambda \in \mathbb{F}_{p^m}^\times$ , an explicit representation for all distinct  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$  is given by a canonical form decomposition for each code, where  $s$  and  $n$  are positive integers satisfying  $\gcd(p, n) = 1$ . For any such code, using its canonical form decomposition the representation for the dual code of the code is provided. Moreover, representations for all distinct negacyclic codes and their dual codes of length  $p^s n$  over  $R$  are obtained, and self-duality for these codes are determined. Finally, all distinct self-dual negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length  $2 \cdot 5^s \cdot 3^t$  are listed for any positive integer  $t$ .

*Keywords:* Constacyclic code; Dual code; Self dual code; Negacyclic code

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## 1. Introduction

Algebraic coding theory deals with the design of error-correcting and error-detecting codes for the reliable transmission of information across noisy channel. The class of constacyclic codes plays a very significant role in the theory of error-correcting codes as they can be efficiently encoded with simple shift registers. This family of codes is thus interesting for both theoretical and practical reasons.

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Let  $\Gamma$  be a commutative finite ring with identity  $1 \neq 0$ , and  $\Gamma^\times$  be the multiplicative group of invertible elements of  $\Gamma$ . For any  $a \in \Gamma$ , we denote by  $\langle a \rangle_\Gamma$ , or  $\langle a \rangle$  for simplicity, the ideal of  $\Gamma$  generated by  $a$ , i.e.,  $\langle a \rangle_\Gamma = a\Gamma = \{ab \mid b \in \Gamma\}$ . For any ideal  $I$  of  $\Gamma$ , we will identify the element  $a + I$  of the residue class ring  $\Gamma/I$  with  $a \pmod{I}$  for any  $a \in \Gamma$  in this paper.

A *code* over  $\Gamma$  of length  $N$  is a nonempty subset  $\mathcal{C}$  of  $\Gamma^N = \{(a_0, a_1, \dots, a_{N-1}) \mid a_j \in \Gamma, j = 0, 1, \dots, N-1\}$ . The code  $\mathcal{C}$  is said to be *linear* if  $\mathcal{C}$  is an  $\Gamma$ -submodule of  $\Gamma^N$ . All codes in this paper are assumed to be linear. The ambient space  $\Gamma^N$  is equipped with the usual Euclidian inner product, i.e.,  $[a, b]_E = \sum_{j=0}^{N-1} a_j b_j$ , where  $a = (a_0, a_1, \dots, a_{N-1}), b = (b_0, b_1, \dots, b_{N-1}) \in \Gamma^N$ , and the *dual code* is defined by  $\mathcal{C}^{\perp_E} = \{a \in \Gamma^N \mid [a, b]_E = 0, \forall b \in \mathcal{C}\}$ . If  $\mathcal{C}^{\perp_E} = \mathcal{C}$ ,  $\mathcal{C}$  is called a *self-dual code* over  $\Gamma$ .

Let  $\gamma \in \Gamma^\times$ . Then a linear code  $\mathcal{C}$  over  $\Gamma$  of length  $N$  is called a  $\gamma$ -*constacyclic code* if  $(\gamma c_{N-1}, c_0, c_1, \dots, c_{N-2}) \in \mathcal{C}$  for all  $(c_0, c_1, \dots, c_{N-1}) \in \mathcal{C}$ . Particularly,  $\mathcal{C}$  is called a *negacyclic code* if  $\gamma = -1$ , and  $\mathcal{C}$  is called a *cyclic code* if  $\gamma = 1$ . For any  $a = (a_0, a_1, \dots, a_{N-1}) \in \Gamma^N$ , let  $a(x) = a_0 + a_1 x + \dots + a_{N-1} x^{N-1} \in \Gamma[x]/\langle x^N - \gamma \rangle$ . We will identify  $a$  with  $a(x)$  in this paper. By [9] Propositions 2.2 and 2.4, we have

**Lemma 1.1** *Let  $\gamma \in \Gamma^\times$ . Then  $\mathcal{C}$  is a  $\gamma$ -constacyclic code of length  $N$  over  $\Gamma$  if and only if  $\mathcal{C}$  is an ideal of the residue class ring  $\Gamma[x]/\langle x^N - \gamma \rangle$ .*

**Lemma 1.2** *The dual code of a  $\gamma$ -constacyclic code of length  $N$  over  $\Gamma$  is a  $\gamma^{-1}$ -constacyclic code of length  $N$  over  $\Gamma$ , i.e., an ideal of  $\Gamma[x]/\langle x^N - \gamma^{-1} \rangle$ .*

In this paper, let  $\mathbb{F}_{p^m}$  be a finite field of cardinality  $p^m$ , where  $p$  is a prime and  $m$  is a positive integer, and denote  $\mathbb{F}_{p^m}[u]/\langle u^2 \rangle$  by  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  ( $u^2 = 0$ ). There were a lot of literatures on linear codes, cyclic codes and constacyclic codes of length  $N$  over rings  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  ( $u^2 = 0$ ) for various prime  $p$  and positive integers  $m$  and  $N$ . For example, [1,2,4,11,13,14 and 17]. The classification of codes plays an important role in studying their structures and encoders. However, it is a very difficult task in general, and only some codes of special lengths over certain finite fields or finite chain rings are classified. For example, all constacyclic codes of length  $2^s$  over the Galois extension rings of  $\mathbb{F}_2 + u\mathbb{F}_2$  are classified and their detailed structures are also established in [8]. In [9], Dinh classified all constacyclic codes of length  $p^s$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ .

Recently, Dinh et al. [10] studied negacyclic codes of length  $2p^s$  over the ring  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ . Chen et al. [7] investigated constacyclic codes of length

$2p^s$  over the ring  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ . The main result and their proofs in [7] and [10] depend heavily on the code length  $2p^s$ , and the papers mainly used the methods in [8] and [9].

From now on, let  $n, s$  be arbitrary positive integers satisfying  $\gcd(p, n) = 1$ , and  $\lambda$  be an arbitrary nonzero element of  $\mathbb{F}_{p^m}$ . In this paper, by use of known results for linear codes over finite chain rings of length 2, we provide a new way different from the methods used in [7]–[10] to determine the algebraic structures, the generators and enumeration of  $\lambda$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ . Specifically, we will address the following questions:

- ◇ Give a precise representation for each  $\lambda$ -constacyclic code  $\mathcal{C}$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ , and provide a simple and clear formula to count the number of codewords in  $\mathcal{C}$ .
- ◇ Give a clear formula to count the number of all  $\lambda$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ .
- ◇ Give a precise representation for the dual code of each  $\lambda$ -constacyclic code  $\mathcal{C}$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ , once the representation of  $\mathcal{C}$  is given.
- ◇ Determine self-dual negacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ .

**Notation 1.3** In the rest of the paper, we denote

- $R = \mathbb{F}_{p^m}[u]/\langle u^2 \rangle = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \ (u^2 = 0)$ ;
- $\mathcal{A} = \mathbb{F}_{p^m}[x]/\langle x^{p^s n} - \lambda \rangle$  and  $\mathcal{A}[u]/\langle u^2 \rangle = \mathcal{A} + u\mathcal{A} \ (u^2 = 0)$ ;
- $\mathcal{R}_\lambda = R[x]/\langle x^{p^s n} - \lambda \rangle$  and  $\mathcal{R}_{\lambda^{-1}} = R[x]/\langle x^{p^s n} - \lambda^{-1} \rangle$ .

The present paper is organized as follows. In Section 2, we provide a direct sum decomposition for any  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$ . Then we determine each direct summand of such decomposition in Section 3. Hence we obtain an explicit representation for each of these codes and give a formula to count the number of codewords in each code from its representation. As a corollary, we obtain a formula to enumerate all such codes. Then we determine the dual code of each code in Section 4. In Section 5, we list all distinct negacyclic codes and their dual codes over  $R$  of length  $p^s n$ , and determine the self-duality for such codes. Finally, for any positive integer  $t$  we list precisely all distinct self-dual negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length  $2 \cdot 5^s \cdot 3^t$  in Section 6.

## 2. Decomposition for $\lambda$ -constacyclic codes over $R$ of length $p^s n$

In this section, we construct a specific ring isomorphism from  $\mathcal{A} + u\mathcal{A}$  ( $u^2 = 0$ ) onto  $\mathcal{R}_\lambda$ . By use of this isomorphism, we obtain a one-to-one correspondence between the set of ideals of  $\mathcal{A} + u\mathcal{A}$  onto the set of ideals of  $\mathcal{R}_\lambda$ , i.e., the set of  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$ .

Let  $g(x) \in \mathcal{R}_\lambda$ . Then  $g(x)$  can be uniquely expressed as

$$g(x) = \sum_{j=0}^{p^s n - 1} g_j x^j, \quad g_j \in R \text{ for } j = 0, 1, \dots, p^s n - 1,$$

where each element  $g_j$  of  $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  is uniquely expressed as

$$g_j = g_{j,0} + u g_{j,1}, \quad g_{j,0}, g_{j,1} \in \mathbb{F}_{p^m}, \quad j = 0, 1, \dots, p^s n - 1.$$

Hence  $g(x) = g_0(x) + u g_1(x)$  where  $g_i(x) = \sum_{j=0}^{p^s n - 1} g_{j,i} x^j \in \mathcal{A}$  for all  $i = 0, 1$ . Moreover, we have the following lemma.

**Lemma 2.1** *For any  $\xi = a(x) + ub(x) \in \mathcal{A} + u\mathcal{A}$ , where  $a(x) = \sum_{j=0}^{p^s n - 1} a_j x^j$  and  $b(x) = \sum_{j=0}^{p^s n - 1} b_j x^j$  with  $a_j, b_j \in \mathbb{F}_{p^m}$ , we define*

$$\Psi(\xi) = a(x) + ub(x) = \sum_{j=0}^{p^s n - 1} (a_j + ub_j) x^j.$$

*Then  $\Psi$  is a ring isomorphism from  $\mathcal{A} + u\mathcal{A}$  onto  $\mathcal{R}_\lambda$ .*

**Proof.** It is clear that  $\Psi$  is bijection from  $\mathcal{A} + u\mathcal{A}$  onto  $\mathcal{R}_\lambda$ . Then by trivial calculations one can verify that  $\Psi(\xi + \eta) = \Psi(\xi) + \Psi(\eta)$  and  $\Psi(\xi \cdot \eta) = \Psi(\xi) \cdot \Psi(\eta)$  for any  $\xi, \eta \in \mathcal{A} + u\mathcal{A}$ .  $\square$

In the rest of this paper, we will identify  $\mathcal{A} + u\mathcal{A}$  with  $\mathcal{R}_\lambda$  under the ring isomorphism  $\Psi$  defined in Lemma 2.1. Then in order to determine all distinct  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$ , it is sufficient to list all distinct ideals of the ring  $\mathcal{A} + u\mathcal{A}$ . To do this, we need to investigate the structure of the ring  $\mathcal{A} = \mathbb{F}_{p^m}[x] / \langle x^{p^s n} - \lambda \rangle$  first.

Since  $\lambda \in \mathbb{F}_{p^m}^\times$  and  $\mathbb{F}_{p^m}^\times$  is a multiplicative cyclic group of order  $p^m - 1$ , there is a unique element  $\lambda_0 \in \mathbb{F}_{p^m}^\times$  such that  $\lambda_0^{p^s} = \lambda$ . From this, we deduce  $x^{p^s n} - \lambda = (x^n - \lambda_0)^{p^s}$  in  $\mathbb{F}_{p^m}[x]$ . As  $\gcd(p, n) = 1$ , there are pairwise coprime monic irreducible polynomials  $f_1(x), \dots, f_r(x)$  in  $\mathbb{F}_{p^m}[x]$  such that  $x^n - \lambda_0 = f_1(x) \dots f_r(x)$ , which implies

$$x^{p^s n} - \lambda = (x^n - \lambda_0)^{p^s} = f_1(x)^{p^s} \dots f_r(x)^{p^s}. \quad (1)$$

For any integer  $j$ ,  $1 \leq j \leq r$ , we assume  $\deg(f_j(x)) = d_j$  and denote  $F_j(x) = \frac{x^n - \lambda_0}{f_j(x)}$ . Then  $F_j(x)^{p^s} = \frac{x^{p^s n} - \lambda}{f_j(x)^{p^s}}$  and  $\gcd(F_j(x), f_j(x)) = 1$ . Hence there exist  $v_j(x), w_j(x) \in \mathbb{F}_{p^m}[x]$  such that  $\deg(v_j(x)) < \deg(f_j(x)) = d_j$  and  $v_j(x)F_j(x) + w_j(x)f_j(x) = 1$ , which implies

$$v_j(x)^{p^s} F_j(x)^{p^s} + w_j(x)^{p^s} f_j(x)^{p^s} = (v_j(x)F_j(x) + w_j(x)f_j(x))^{p^s} = 1. \quad (2)$$

In the rest of this paper, we adopt the following notations.

**Notation 2.2** Let  $1 \leq j \leq r$ . Using the notations above, we denote

$$\mathcal{K}_j = \mathbb{F}_{p^m}[x]/\langle f_j(x)^{p^s} \rangle, \mathcal{K}_j[u]/\langle u^2 \rangle = \mathcal{K}_j + u\mathcal{K}_j \ (u^2 = 0)$$

and set

$$\begin{aligned} \theta_j(x) &\equiv v_j(x)F_j(x) = 1 - w_j(x)f_j(x) \pmod{x^n - \lambda_0}, \\ \varepsilon_j(x) &= \theta_j(x)^{p^s} \equiv v_j(x)^{p^s} F_j(x)^{p^s} = 1 - w_j(x)^{p^s} f_j(x)^{p^s} \pmod{x^{p^s n} - \lambda}. \end{aligned} \quad (3)$$

Then by Equations (1)–(3) and the Chinese remainder theorem for commutative rings with identity, we deduce the following conclusion.

**Lemma 2.3** *Using the notations above, we have the following:*

(i)  $\varepsilon_1(x) + \dots + \varepsilon_r(x) = 1$ ,  $\varepsilon_j(x)^2 = \varepsilon_j(x)$  and  $\varepsilon_j(x)\varepsilon_l(x) = 0$  in the ring  $\mathcal{A}$  for all  $1 \leq j \neq l \leq r$ .

(ii)  $\mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_r$  where  $\mathcal{A}_j = \mathcal{A}\varepsilon_j(x)$  with  $\varepsilon_j(x)$  as its multiplicative identity and satisfies  $\mathcal{A}_j\mathcal{A}_l = \{0\}$  for all  $1 \leq j \neq l \leq r$ .

(iii) For any integer  $j$ ,  $1 \leq j \leq r$ , for any  $a(x) \in \mathcal{K}_j$  we define

$$\varphi_j : a(x) \mapsto \varepsilon_j(x)a(x) \pmod{x^{p^s n} - \lambda}.$$

Then  $\varphi_j$  is a ring isomorphism from  $\mathcal{K}_j$  onto  $\mathcal{A}_j$ .

(iv) For any  $a_j(x) \in \mathcal{K}_j$  for  $j = 1, \dots, r$ , define

$$\varphi(a_1(x), \dots, a_r(x)) = \sum_{j=1}^r \varphi_j(a_j(x)) = \sum_{j=1}^r \varepsilon_j(x)a_j(x) \pmod{x^{p^s n} - \lambda}.$$

Then  $\varphi$  is a ring isomorphism from  $\mathcal{K}_1 \times \dots \times \mathcal{K}_r$  onto  $\mathcal{A}$ .

Next, we investigate the structure of the ring  $\mathcal{A} + u\mathcal{A}$ .

**Lemma 2.4** *Using the notations in Lemma 2.3, for any  $\xi_j + u\eta_j \in \mathcal{K}_j + u\mathcal{K}_j$  with  $\xi_j, \eta_j \in \mathcal{K}_j$ , where  $j = 1, \dots, r$ , we define*

$$\Phi(\xi_1 + u\eta_1, \dots, \xi_r + u\eta_r) = \sum_{j=1}^r (\varphi_j(\xi_j) + u\varphi_j(\eta_j)) = \sum_{j=1}^r \varepsilon_j(x)(\xi_j + u\eta_j). \quad (4)$$

Then  $\Phi$  is a ring isomorphism from  $(\mathcal{K}_1 + u\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + u\mathcal{K}_r)$  onto  $\mathcal{A} + u\mathcal{A}$ .

**Proof.** By Lemma 2.3(iv), it is clear that the ring isomorphism  $\varphi : \mathcal{K}_1 \times \dots \times \mathcal{K}_r \rightarrow \mathcal{A}$  can be extended to a polynomial ring isomorphism  $\Phi_0$  from  $\mathcal{K}_1[u] \times \dots \times \mathcal{K}_r[u]$  onto  $\mathcal{A}[u]$  by the rule that

$$\Phi_0 \left( \sum_t \xi_{1,t} u^t, \dots, \sum_t \xi_{r,t} u^t \right) = \sum_t \left( \sum_{j=1}^r \varphi_j(\xi_{j,t}) \right) u^t \quad (\forall \xi_{j,t} \in \mathcal{K}_j).$$

Then by classical ring theory, we see that  $\Phi_0$  induces a residue class ring isomorphism  $\Phi$  from  $(\mathcal{K}_1[u]/\langle u^2 \rangle) \times \dots \times (\mathcal{K}_r[u]/\langle u^2 \rangle)$  onto  $\mathcal{A}[u]/\langle u^2 \rangle$  define by (4). Now, the conclusion follows from Notation 2.2 and  $\mathcal{A}[u]/\langle u^2 \rangle = \mathcal{A} + u\mathcal{A}$  ( $u^2 = 0$ ), immediately.  $\square$

Finally, we give a direct sum decomposition for any  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$ .

**Theorem 2.5** *Using the notations above, let  $\mathcal{C} \subseteq \mathcal{R}_\lambda = R[x]/\langle x^{p^s n} - \lambda \rangle$ . Then the following statements are equivalent:*

- (i)  $\mathcal{C}$  is a  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$ , i.e., an ideal of  $\mathcal{R}_\lambda$ ;
- (ii)  $\mathcal{C}$  is an ideal of  $\mathcal{A} + u\mathcal{A}$ ;
- (iii) For each integer  $j$ ,  $1 \leq j \leq r$ , there is a unique ideal  $C_j$  of  $\mathcal{K}_j + u\mathcal{K}_j$  such that  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x) C_j \pmod{x^{p^s n} - \lambda}$ .

**Proof.** (i) $\Leftrightarrow$ (ii) It follows from that  $\mathcal{A} + u\mathcal{A} = \mathcal{R}_\lambda$  under a ring isomorphism.

(ii) $\Leftrightarrow$ (iii) By Lemma 3.5 we know that  $\mathcal{C}$  is an ideal of  $\mathcal{A} + u\mathcal{A}$  if and only if there is a unique ideal  $I$  of the ring  $(\mathcal{K}_1 + u\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + u\mathcal{K}_r)$  such that  $\Phi(I) = \mathcal{C}$ . Furthermore, by classical ring theory we see that  $I$  is an ideal of  $(\mathcal{K}_1 + u\mathcal{K}_1) \times \dots \times (\mathcal{K}_r + u\mathcal{K}_r)$  if and only if for each integer  $j$ ,  $1 \leq j \leq r$ , there is a unique ideal  $C_j$  of  $\mathcal{K}_j + u\mathcal{K}_j$  such that

$$I = C_1 \times \dots \times C_r = \{(\alpha_1, \dots, \alpha_r) \mid \alpha_j \in C_j, j = 1, \dots, r\}.$$

When this condition is satisfied, by Equation (5) we have

$$\begin{aligned} \mathcal{C} &= \Phi(I) = \{\Phi(\alpha_1, \dots, \alpha_r) \mid \alpha_j \in C_j, j = 1, \dots, r\} \\ &= \left\{ \sum_{j=1}^r \varepsilon_j(x) \alpha_j \mid \alpha_j \in C_j, j = 1, \dots, r \right\}. \end{aligned}$$

Hence  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x)C_j$ , since  $\varepsilon_j(x)C_j = \{\varepsilon_j(x)\alpha_j \mid \alpha_j \in C_j\}$  for all  $j$ .  $\square$

Therefore, in order to determine all distinct  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$ , by Theorem 2.5 it is sufficient to list all distinct ideals of the ring  $\mathcal{K}_j + u\mathcal{K}_j$  ( $u^2 = 0$ ) for all  $j = 1, \dots, r$ .

### 3. Representation for ideals of $\mathcal{K}_j + u\mathcal{K}_j$

In this section, we determine all distinct ideals of  $\mathcal{K}_j + u\mathcal{K}_j$  for all  $j = 1, \dots, r$ . As  $\mathcal{K}_j = \mathbb{F}_{p^m}[x]/\langle f_j(x)^{p^s} \rangle$  where  $f_j(x)$  is a monic irreducible polynomial in  $\mathbb{F}_{p^m}[x]$  of degree  $d_j$ , we have the following conclusions.

**Lemma 3.1** (cf. [5] Lemma 3.7 and [6] Example 2.1)  *$\mathcal{K}_j$  have the following properties:*

(i)  *$\mathcal{K}_j$  is a finite chain ring,  $f_j(x)$  generates the unique maximal ideal  $\langle f_j(x) \rangle$  of  $\mathcal{K}_j$ , the nilpotency index of  $f_j(x)$  is equal to  $p^s$  and the residue class field of  $\mathcal{K}_j$  modulo  $\langle f_j(x) \rangle$  is  $\mathcal{K}_j/\langle f_j(x) \rangle \cong \mathbb{F}_{p^m}[x]/\langle f_j(x) \rangle$ , where  $\mathbb{F}_{p^m}[x]/\langle f_j(x) \rangle$  is an extension field of  $\mathbb{F}_{p^m}$  with  $p^{md_j}$  elements.*

(ii) *Let  $\mathcal{T}_j = \{\sum_{i=0}^{d_j-1} t_i x^i \mid t_0, t_1, \dots, t_{d_j-1} \in \mathbb{F}_{p^m}\}$ . Then  $|\mathcal{T}_j| = p^{md_j}$ , and every element  $\xi$  of  $\mathcal{K}_j$  has a unique  $f_j(x)$ -adic expansion:*

$$\xi = \sum_{k=0}^{p^s-1} b_k(x)f(x)^k, \text{ where } b_k(x) \in \mathcal{T}_j \text{ for all } k = 0, 1, \dots, p^s - 1.$$

*If  $\xi \neq 0$ , the  $f_j(x)$ -degree of  $\xi$  is defined as the least index  $k$  for which  $b_k(x) \neq 0$  and denoted as  $\|\xi\|_{f_j(x)} = k$ . If  $\xi = 0$  we write  $\|\xi\|_{f_j(x)} = p^s$ . Then  $\xi \in \mathcal{K}_j^\times$  if and only if  $\|\xi\|_{f_j(x)} = 0$ .*

(iii) *All distinct ideals of  $\mathcal{K}_j$  are given by:  $\langle f_j(x)^l \rangle = f_j(x)^l \mathcal{K}_j$ ,  $l = 0, 1, \dots, p^s$ . Moreover,  $|\langle f_j(x)^l \rangle| = p^{md_j(p^s-l)}$  for  $l = 0, 1, \dots, p^s$*

(iv) *Let  $1 \leq l \leq p^s$ . Then  $\mathcal{K}_j/\langle f_j(x)^l \rangle = \{\sum_{k=0}^{l-1} b_k(x)f(x)^k \mid b_k(x) \in \mathcal{T}_j, k = 0, 1, \dots, l-1\}$  and hence  $|\mathcal{K}_j/\langle f_j(x)^l \rangle| = p^{md_j l}$ .*

(v) *For any  $0 \leq l \leq t \leq p^s - 1$ , we have*

$$f_j(x)^l(\mathcal{K}_j/\langle f_j(x)^t \rangle) = \{\sum_{k=l}^{t-1} b_k(x)f(x)^k \mid b_k(x) \in \mathcal{T}_j, k = l, \dots, t-1\}$$

*and  $|f_j(x)^l(\mathcal{K}_j/\langle f_j(x)^t \rangle)| = p^{md_j(t-l)}$ , where we set  $f_j(x)^l(\mathcal{K}_j/\langle f_j(x)^l \rangle) = \{0\}$  for convenience.*

By Notation 2.2, the addition and multiplication on the ring  $\mathcal{K}_j + u\mathcal{K}_j$  are defined by: for any  $\xi_0, \xi_1, \eta_0, \eta_1 \in \mathcal{K}_j$ ,

$$\diamond (\xi_0 + u\xi_1) + (\eta_0 + u\eta_1) = (\xi_0 + \eta_0) + u(\xi_1 + \eta_1);$$

$$\diamond (\xi_0 + u\xi_1)(\eta_0 + u\eta_1) = \xi_0\eta_0 + u(\xi_0\eta_1 + \xi_1\eta_0).$$

Hence  $\mathcal{K}_j$  is a subring of  $\mathcal{K}_j + u\mathcal{K}_j$ . Furthermore,  $\mathcal{K}_j + u\mathcal{K}_j$  is a  $\mathcal{K}_j$ -module with a  $\mathcal{K}_j$ -basis  $\{1, u\}$ . Now, we define

$$\varsigma : \mathcal{K}_j^2 \rightarrow \mathcal{K}_j + u\mathcal{K}_j \text{ via } \varsigma : (a_0, a_1) \mapsto a_0 + ua_1 \ (\forall a_0, a_1 \in \mathcal{K}_j).$$

One can easily verify that  $\varsigma$  is a  $\mathcal{K}_j$ -module isomorphism from  $\mathcal{K}_j^2$  onto  $\mathcal{K}_j + u\mathcal{K}_j$ . Using this  $\mathcal{K}_j$ -module isomorphism  $\varsigma$ , we can determine ideals of the ring  $\mathcal{K}_j + u\mathcal{K}_j$  from  $\mathcal{K}_j$ -submodules of  $\mathcal{K}_j^2$  satisfying certain conditions.

**Lemma 3.2** *Using the notations above,  $C$  is an ideal of the ring  $\mathcal{K}_j + u\mathcal{K}_j$  if and only if there is a unique  $\mathcal{K}_j$ -submodule  $S$  of  $\mathcal{K}_j^2$  satisfying the following condition:*

$$(0, a_0) \in S, \ \forall (a_0, a_1) \in S \tag{5}$$

such that  $C = \varsigma(S)$ .

**Proof.** Let  $C$  be an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$ . Since  $\mathcal{K}_j$  is a subring of  $\mathcal{K}_j + u\mathcal{K}_j$ , we see that  $C$  is a  $\mathcal{K}_j$ -submodule of  $\mathcal{K}_j + u\mathcal{K}_j$  satisfying  $u\xi \in C$  for any  $\xi \in C$ . Now, let  $S = \{(a_0, a_1) \mid a_0 + ua_1 \in C\} = \varsigma^{-1}(C)$ . It is obvious that  $S$  is a  $\mathcal{K}_j$ -submodule of  $\mathcal{K}_j^2$  satisfying  $C = \varsigma(S)$ . Moreover, for any  $(a_0, a_1) \in S$ , i.e.  $a_0 + ua_1 \in C$ , by  $u^2 = 0$  it follows that  $ua_0 = u(a_0 + ua_1) \in C$ . Hence  $(0, a_0) \in S$ .

Conversely, let  $C = \varsigma(S)$  and  $S$  be a  $\mathcal{K}_j$ -submodule of  $\mathcal{K}_j^2$  satisfying the condition required. For any  $b_0, b_1 \in \mathcal{K}_j$  and  $a_0 + ua_1 \in C$  where  $(a_0, a_1) \in S$ , by  $(0, a_0) \in S$  we have  $b_0(a_0, a_1) + b_1(0, a_0) \in S$ . On the other hand, we have  $(b_0 + ub_1)(a_0 + ua_1) = b_0a_0 + u(b_0a_1 + b_1a_0)$  in  $\mathcal{K}_j + u\mathcal{K}_j$ , which implies  $(b_0 + ub_1)(a_0 + ua_1) = \varsigma(b_0a_0, b_0a_1 + b_1a_0) = \varsigma(b_0(a_0, a_1) + b_1(0, a_0)) \in \varsigma(S) = C$ . Hence  $C$  is an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$ .  $\square$

We notice that  $\mathcal{K}_j$ -submodules of  $\mathcal{K}_j^2$  are called linear codes over the finite chain ring  $\mathcal{K}_j$  of length 2. Let  $S$  be a linear code over  $\mathcal{K}_j$  of length 2. By [15] Definition 3.1, a matrix  $G$  is called a *generator matrix* for  $S$  if every codeword in  $S$  is an  $\mathcal{K}_j$ -linear combination of the row vectors of  $G$  and any row vector of  $G$  can not be written as an  $\mathcal{K}_j$ -linear combination of the other row vectors of  $G$ . For linear codes over  $\mathcal{K}_j$  of length 2 and their generator matrices, we list the following lemmas.



**Lemma 3.3** (cf. [6] Lemma 2.2) *The number of linear codes over the finite chain ring  $\mathcal{K}_j$  of length 2 is equal to  $\sum_{j=0}^{p^s} (2j+1) |\mathcal{K}_j / \langle f_j(x) \rangle|^{p^s-j} = \sum_{j=0}^{p^s} (2j+1) p^{m(p^s-j)d_j}$ .*

**Lemma 3.4** (cf. [6] Example 2.5) *Every linear code  $S$  over  $\mathcal{K}_j$  of length 2 has one and only one of the following matrices  $G$  as their generator matrices:*

- (i)  $G = (1, a(x))$ ,  $a(x) \in \mathcal{K}_j$ .
- (ii)  $G = (f_j(x)^k, f_j(x)^k a(x))$ ,  $a(x) \in \mathcal{K}_j / \langle f_j(x)^{p^s-k} \rangle$  and  $1 \leq k \leq p^s - 1$ .
- (iii)  $G = (f_j(x)b(x), 1)$ ,  $b(x) \in \mathcal{K}_j / \langle f_j(x)^{p^s-1} \rangle$ .
- (iv)  $G = (f_j(x)^{k+1}b(x), f_j(x)^k)$ ,  $b(x) \in \mathcal{K}_j / \langle f_j(x)^{p^s-k-1} \rangle$  and  $1 \leq k \leq p^s - 1$ .
- (v)  $G = \begin{pmatrix} f_j(x)^k & 0 \\ 0 & f_j(x)^k \end{pmatrix}$ ,  $0 \leq k \leq p^s$ .
- (vi)  $G = \begin{pmatrix} 1 & c(x) \\ 0 & f_j(x)^t \end{pmatrix}$ ,  $c(x) \in \mathcal{K}_j / \langle f_j(x)^t \rangle$  and  $1 \leq t \leq p^s - 1$ .
- (vii)  $G = \begin{pmatrix} f_j(x)^k & f_j(x)^k c(x) \\ 0 & f_j(x)^{k+t} \end{pmatrix}$ ,  $c(x) \in \mathcal{K}_j / \langle f_j(x)^t \rangle$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .
- (viii)  $G = \begin{pmatrix} c(x) & 1 \\ f_j(x)^t & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)(\mathcal{K}_j / \langle f_j(x)^t \rangle)$  and  $1 \leq t \leq p^s - 1$ .
- (ix)  $G = \begin{pmatrix} f_j(x)^k c(x) & f_j(x)^k \\ f_j(x)^{k+t} & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)(\mathcal{K}_j / \langle f_j(x)^t \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

For any vector  $(\xi_0, \xi_1) \in \mathcal{K}_j^2$ , we define the  $f_j(x)$ -degree of  $(\xi_0, \xi_1)$  by  $\|(\xi_0, \xi_1)\|_{f_j(x)} = \min\{\|\xi_0\|_{f_j(x)}, \|\xi_1\|_{f_j(x)}\}$ .

**Lemma 3.5** (cf. [15] Proposition 3.2 and Theorem 3.5) *Let  $S$  be a nonzero linear code over  $\mathcal{K}_j$  of length 2, and  $G$  be a generator matrix of  $S$  with row vectors  $G_1, \dots, G_\rho$  satisfying  $\|G_j\|_{f_j(x)} = t_j$ , where  $0 \leq t_1 \leq \dots \leq t_\rho \leq p^s - 1$ .*

- (i) *Every codeword in  $S$  can be uniquely expressed as  $\sum_{j=1}^{\rho} b_j(x) G_j$  with  $b_j(x) \in \mathcal{K}_j / \langle f_j(x)^{p^s-t_j} \rangle$  for all  $j = 1, \dots, \rho$ .*
- (ii) *The number of codewords in  $S$  is equal to*

$$|S| = |\mathcal{K}_j / \langle f_j(x) \rangle|^{\sum_{j=1}^{\rho} (p^s-t_j)} = |\mathcal{T}_j|^{\sum_{j=1}^{\rho} (p^s-t_j)} = p^{md_j \sum_{j=1}^{\rho} (p^s-t_j)}.$$

The following lemma will be needed in the proof of Lemma 3.7.

**Lemma 3.6** (cf. [16] Lemma 2.2 and Corollary 2.3]) *Using the notations above, we have the following conclusions.*

- (i) *For any  $\xi \in \mathcal{K}_j$  with  $\xi \neq 0$ , there is a unique integer  $k$ ,  $0 \leq k \leq p^s - 1$ , such that  $\xi = f_j(x)^k c(x)$  for some  $c(x) \in \mathcal{K}_j^\times$ . In this case,  $\|\mathcal{K}_j\|_{f_j(x)} = k$  and  $c(x)$  is unique modulo  $f_j(x)^{p^s-k}$ , i.e.,  $c(x) \in (\mathcal{K}_j / \langle f_j(x)^{p^s-k} \rangle)^\times$ .*
- (ii) *Let  $1 \leq t \leq l \leq p^s$  and  $\xi \in \mathcal{K}_j$ . Then  $f_j(x)^t \xi \in f_j(x)^l \mathcal{K}_j$  if and only if  $\|\xi\|_{f_j(x)} \geq l - t$ , i.e.,  $\xi \in f_j(x)^{l-t} \mathcal{K}_j$ .*

Then we determine linear codes over  $\mathcal{K}_j$  of length 2, i.e.,  $\mathcal{K}_j$ -submodules of  $\mathcal{K}_j^2$ , satisfying Condition (5) in Lemma 3.2.

**Lemma 3.7** *Using the notations above, every linear code  $S$  over  $\mathcal{K}_j$  of length 2 satisfying Condition (5) in Lemma 3.2 has one and only one of the following matrices  $G$  as their generator matrices:*

- (I)  $G = (f_j(x)b(x), 1)$ ,  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s-1} \rangle)$ .
- (II)  $G = (f_j(x)^{k+1}b(x), f_j(x)^k)$ ,  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-k-2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s-k-1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .
- (III)  $G = \begin{pmatrix} f_j(x)^k & 0 \\ 0 & f_j(x)^k \end{pmatrix}$ ,  $0 \leq k \leq p^s$ .
- (IV)  $G = \begin{pmatrix} c(x) & 1 \\ f_j(x)^t & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)^{\lceil \frac{t}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^t \rangle)$  and  $1 \leq t \leq p^s - 1$ .
- (V)  $G = \begin{pmatrix} f_j(x)^k c(x) & f_j(x)^k \\ f_j(x)^{k+t} & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)^{\lceil \frac{t}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^t \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

**Proof.** For notations simplicity, we denote  $\pi = f_j(x)$  and use lowercase letters, say  $a, b, c, \dots$ , to denote elements of the finite chain ring  $\mathcal{K}_j$ . By Lemma 3.4 we only need to consider the following nine cases.

- (i)  $G = (1, a)$ , where  $a \in \mathcal{K}_j$ . Suppose that  $S$  satisfies Condition (5). Then  $(0, 1) \in S$ . Since  $G$  is the generator matrix of  $S$ , there exists  $b \in \mathcal{K}_j$  such that  $(0, 1) = b(1, a) = (b, ba)$ , i.e.,  $0 = b$  and  $1 = ba$ , which is impossible. Hence  $S$  does not satisfy Condition (5) in this case.

- (ii)  $G = (\pi^k, \pi^k a)$ , where  $a \in \mathcal{K}_j / \langle \pi^{p^s-k} \rangle$  and  $1 \leq k \leq p^s - 1$ . Suppose

that  $S$  satisfies Condition (5). Then  $(0, \pi^k) \in S$ . So there exists  $b \in \mathcal{K}_j$  such that  $(0, \pi^k) = b(\pi^k, \pi^k a) = (\pi^k b, \pi^k ab)$ , which implies  $0 = \pi^k b$  and  $\pi^k = \pi^k ba$ . Hence  $\pi^k = 0$ . But  $1 \leq k \leq p^s - 1$ , we get a contradiction. Hence  $S$  does not satisfy Condition (5) in this case.

(iii)  $G = (\pi b, 1)$ , where  $b \in \mathcal{K}_j / \langle \pi^{p^s-1} \rangle$ . Then  $S$  satisfies Condition (5) if and only if there exists  $a \in \mathcal{K}_j$  such that  $(0, \pi b) = a(\pi b, 1) = (\pi ba, a)$ , i.e.,  $0 = \pi ba$  and  $\pi b = a$ , which are equivalent to that  $b$  satisfies  $\pi^2 b^2 = 0$ . By Lemma 3.6(ii) we see that  $b \in \mathcal{K}_j / \langle \pi^{p^s-1} \rangle$  satisfying  $\pi^2 b^2 = 0$  if and only if  $b^2 \in \pi^{p^s-2} A$ , i.e.,  $2\|b\|_\pi = \|b^2\|_\pi \geq p^s - 2$ , and hence  $\|b\|_\pi \geq \lceil \frac{1}{2}(p^s - 2) \rceil$  where  $\lceil \frac{1}{2}(p^s - 2) \rceil = \min\{l \in \mathbb{Z}^+ \cup \{0\} \mid l \geq \frac{1}{2}(p^s - 2)\}$ . Therefore,  $b \in \pi^{\lceil \frac{1}{2}(p^s-2) \rceil} \mathcal{K}_j \cap (\mathcal{K}_j / \langle \pi^{p^s-1} \rangle) = \pi^{\lceil \frac{1}{2}(p^s-2) \rceil} (\mathcal{K}_j / \langle \pi^{p^s-1} \rangle)$ .

(iv)  $G = (\pi^{k+1}b, \pi^k)$ , where  $b \in \mathcal{K}_j / \langle \pi^{p^s-k-1} \rangle$  and  $1 \leq k \leq p^s - 1$ . Then  $S$  satisfies Condition (5) if and only if there exists  $a \in \mathcal{K}_j$  such that  $(0, \pi^{k+1}b) = a(\pi^{k+1}b, \pi^k) = (\pi^{k+1}ab, \pi^k a)$ , i.e.,  $0 = \pi^{k+1}ab$  and  $\pi^{k+1}b = \pi^k a$ , which are equivalent to that  $b$  satisfies  $\pi^{k+2}b^2 = 0$ . Then by  $3 \leq k+2 \leq p^s+1$ , we have one of the following two subcases:

(iv-1) When  $k+2 \geq p^s$ , i.e.,  $k = p^s - 2$  or  $p^s - 1$ , then  $\pi^{k+2} = 0$  and hence  $\pi^{k+2}b^2 = 0$  for every  $b \in \mathcal{K}_j / \langle \pi^{p^s-k-1} \rangle$ .

(iv-2) When  $k+2 \leq p^s - 1$ , i.e.,  $k \leq p^s - 3$  (and  $p^s \geq 4$ ), then  $b \in \mathcal{K}_j / \langle \pi^{p^s-k-1} \rangle$  satisfying  $\pi^{k+2}b^2 = 0$  if and only if  $b^2 \in \pi^{p^s-k-2} \mathcal{K}_j$ . From this and by Lemma 3.6(ii), we deduce that  $\|b\|_\pi \geq \lceil \frac{1}{2}(p^s - k - 2) \rceil$ . Therefore,  $b \in \pi^{\lceil \frac{1}{2}(p^s-k-2) \rceil} (\mathcal{K}_j / \langle \pi^{p^s-k-1} \rangle)$ .

(v)  $G = \begin{pmatrix} \pi^k & 0 \\ 0 & \pi^k \end{pmatrix}$ , where  $0 \leq k \leq p^s$ . It is clear that  $S$  satisfies Condition (5) for any  $k$ .

(vi)  $G = \begin{pmatrix} 1 & c \\ 0 & \pi^t \end{pmatrix}$ , where  $c \in \mathcal{K}_j / \langle \pi^t \rangle$  and  $1 \leq t \leq p^s - 1$ . Suppose that  $S$  satisfies Condition (5). Then there exist  $a, b \in \mathcal{K}_j$  such that  $(0, 1) = a(1, c) + b(0, \pi^t) = (a, ac + \pi^t b)$ , i.e.,  $0 = a$  and  $1 = ac + \pi^t b$ , which implies  $1 = \pi^t b$ , and we get a contradiction. Hence  $S$  does not satisfy Condition (5) in this case.

(vii)  $G = \begin{pmatrix} \pi^k & \pi^k c \\ 0 & \pi^{k+t} \end{pmatrix}$ , where  $c \in \mathcal{K}_j / \langle \pi^t \rangle$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ . Suppose that  $S$  satisfies Condition (5). Then there exist  $a, b \in \mathcal{K}_j$  such that  $(0, \pi^k) = a(\pi^k, \pi^k c) + b(0, \pi^{k+t}) = (\pi^k a, \pi^k ac + \pi^{k+t} b)$ , i.e.,  $0 = \pi^k a$  and  $\pi^k = \pi^k ac + \pi^{k+t} b$ , which implies  $\pi^k = \pi^{k+t} b$ , and we get a

contradiction. Hence  $S$  does not satisfy Condition (5) in this case.

(viii)  $G = \begin{pmatrix} c & 1 \\ \pi^t & 0 \end{pmatrix}$ , where  $c \in \pi(\mathcal{K}_j/\langle \pi^t \rangle)$  and  $1 \leq t \leq p^s - 1$ . It is clear that  $(0, \pi^t) = \pi^t(c, 1) - c(\pi^t, 0) \in S$ . Then  $S$  satisfies Condition (5) if and only if there exist  $a, b \in \mathcal{K}_j$  such that  $(0, c) = a(c, 1) + b(\pi^t, 0) = (ac + \pi^t b, a)$ , i.e.,  $0 = ac + \pi^t b$  and  $c = a$ , which are equivalent to that  $c^2 = -\pi^t b \in \pi^t \mathcal{K}_j$ , i.e.,  $2\|c\|_\pi \geq t$ . Therefore,  $c \in \pi^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j/\langle \pi^t \rangle)$  by Lemma 3.6(ii).

(ix)  $G = \begin{pmatrix} \pi^k c & \pi^k \\ \pi^{k+t} & 0 \end{pmatrix}$ , where  $c \in \pi(\mathcal{K}_j/\langle \pi^t \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ . Obviously,  $(0, \pi^{k+t}) = \pi^t(\pi^k c, \pi^k) - c(\pi^{k+t}, 0) \in S$ . Then  $S$  satisfies Condition (5) if and only if there exist  $a, b \in \mathcal{K}_j$  such that  $(0, \pi^k c) = a(\pi^k c, \pi^k) + b(\pi^{k+t}, 0) = (\pi^k ac + \pi^{k+t} b, \pi^k a)$ , i.e.,  $0 = \pi^k ac + \pi^{k+t} b$  and  $\pi^k c = \pi^k a$ , which are equivalent to that  $\pi^k c^2 = -\pi^{k+t} b \in \pi^{k+t} \mathcal{K}_j$ . Then by Lemma 3.6(ii) we deduce that  $c^2 \in \pi^t \mathcal{K}_j$ . Hence  $c \in \pi^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j/\langle \pi^t \rangle)$ .  $\square$

Now, we can list all distinct ideals of the ring  $\mathcal{K}_j + u\mathcal{K}_j$ .

**Theorem 3.8** *Using the notations above, all distinct ideals  $C_j$  of the ring  $\mathcal{K}_j + u\mathcal{K}_j$  ( $u^2 = 0$ ) are given by one of the following five cases:*

(I)  $p^{(p^s-1-\lceil \frac{1}{2}(p^s-2) \rceil)md_j}$  ideals:

$$C_j = \langle f_j(x)b(x) + u \rangle \text{ with } |C_j| = p^{md_j p^s},$$

where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-2) \rceil}(\mathcal{K}_j/\langle f_j(x)^{p^s-1} \rangle)$ .

(II)  $\sum_{k=1}^{p^s-1} p^{(p^s-k-1-\lceil \frac{1}{2}(p^s-k-2) \rceil)md_j}$  ideals:

$$C_j = \langle f_j(x)^{k+1}b(x) + u f_j(x)^k \rangle \text{ with } |C_j| = p^{md_j(p^s-k)},$$

where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-k-2) \rceil}(\mathcal{K}_j/\langle f_j(x)^{p^s-k-1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .

(III)  $p^s + 1$  ideals:  $C_j = \langle f_j(x)^k \rangle$  with  $|C_j| = p^{2md_j(p^s-k)}$ ,  $0 \leq k \leq p^s$ .

(IV)  $\sum_{t=1}^{p^s-1} p^{(t-\lceil \frac{t}{2} \rceil)md_j}$  ideals:

$$C_j = \langle f_j(x)b(x) + u, f_j(x)^t \rangle \text{ with } |C_j| = p^{md_j(2p^s-t)},$$

where  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_j/\langle f_j(x)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - 1$ .

(V)  $\sum_{k=1}^{p^s-2} \sum_{t=1}^{p^s-k-1} p^{(t-\lceil \frac{t}{2} \rceil)md_j}$  ideals:

$$C_j = \langle f_j(x)^{k+1}b(x) + uf_j(x)^k, f_j(x)^{k+t} \rangle \text{ with } |C_j| = p^{md_j(2p^s-2k-t)},$$

where  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

Therefore, the number of ideals of  $\mathcal{K}_j + u\mathcal{K}_j$  is equal to

$$N_{(p^m, d_j, p^s)} = 1 + p^s + \sum_{k=0}^{p^s-1} p^{(p^s-k-1-\lceil \frac{1}{2}(p^s-k-2) \rceil)md_j} + \sum_{k=0}^{p^s-2} \sum_{t=1}^{p^s-k-1} p^{(t-\lceil \frac{t}{2} \rceil)md_j}.$$

**Proof.** Let  $C_j$  be a nontrivial ideal of  $\mathcal{K}_j + u\mathcal{K}_j$ . By Lemma 3.2, there is a unique  $\mathcal{K}_j$ -submodule  $S_j$  of  $\mathcal{K}_j^2$  satisfying Condition (5):  $(0, a_0) \in S_j$  for all  $(a_0, a_1) \in S_j$ , such that  $C_j = \varsigma(S_j)$ . By Lemma 3.7, we see that  $S_j$  has one and only one of the following matrices  $G_j$  as their generator matrices:

(i)  $G_j = (f_j(x)b(x), 1)$ ,  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{p^s-1} \rangle)$ .

(ii)  $G_j = (f_j(x)^{k+1}b(x), f_j(x)^k)$ ,  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s-k-2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{p^s-k-1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .

(iii)  $G_j = \begin{pmatrix} f_j(x)^k & 0 \\ 0 & f_j(x)^k \end{pmatrix}$ ,  $0 \leq k \leq p^s$ .

(iv)  $G_j = \begin{pmatrix} c(x) & 1 \\ f_j(x)^t & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^t \rangle)$  and  $1 \leq t \leq p^s - 1$ .

(v)  $G_j = \begin{pmatrix} f_j(x)^k c(x) & f_j(x)^k \\ f_j(x)^{k+t} & 0 \end{pmatrix}$ ,  $c(x) \in f_j(x)^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^t \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

(I) Let  $G_j$  be given in (i). By Lemma 3.2 we have  $C_j = \varsigma(S_j) = \langle \varsigma(f_j(x)b(x), 1) \rangle = \langle f_j(x)b(x) + u \rangle$ . As  $\|(f_j(x)b(x), 1)\|_{f_j(x)} = 0$ , by Lemma 3.5(ii) the number of elements in  $S_j$  is equal to  $|S_j| = p^{md_j(p^s-0)} = p^{md_j p^s}$ . Hence  $|C_j| = |S_j| = p^{md_j p^s}$  by Lemma 3.2.

In this case, by Lemma 3.1(v) and Lemma 3.2 we deduce that the number of ideals is equal to  $|f_j(x)^{\lceil \frac{1}{2}(p^s-2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{p^s-1} \rangle)| = p^{md_j(p^s-1-\lceil \frac{1}{2}(p^s-2) \rceil)}$ .

Case (II) can be proved similarly as that of (I).

(III) Let  $G_j$  be given in (iii). By Lemma 3.2, it follows that

$$C_j = \varsigma(S_j) = \langle \varsigma(f_j(x)^k, 0), \varsigma(0, f_j(x)^k) \rangle = \langle f_j(x)^k, uf_j(x)^k \rangle = \langle f_j(x)^k \rangle.$$

As  $\|(f_j(x)^k, 0)\|_{f_j(x)} = \|(0, f_j(x)^k)\|_{f_j(x)} = k$ , by Lemmas 3.2 and 3.5 we deduce that  $|C_j| = |S_j| = p^{md_j((p^s-k)+(p^s-k))} = p^{2md_j(p^s-k)}$ .

(IV) Let  $G_j$  be given in (iv). By Lemma 3.2 we have  $C_j = \varsigma(S_j) = \langle \varsigma(c(x), 1), \varsigma(f_j(x)^t, 0) \rangle = \langle c(x) + u, f_j(x)^t \rangle$ . As  $\|(c(x), 1)\|_{f_j(x)} = 0$  and  $\|(f_j(x)^t, 0)\|_{f_j(x)} = t$ , by Lemmas 3.2 and 3.5 we deduce that  $|C_j| = |S_j| = p^{md_j((p^s-0)+(p^s-t))} = p^{md_j(2p^s-t)}$ .

In this case, by Lemma 3.1(v) and Lemma 3.2 we deduce that the number of ideals is equal to  $|f_j(x)^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^t \rangle)| = p^{md_j(t - \lceil \frac{t}{2} \rceil)}$ .

Furthermore, by  $\lceil \frac{t}{2} \rceil \geq 1$  we can write  $c(x) = f_j(x)b(x)$  for any  $c(x) \in f_j(x)^{\lceil \frac{t}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^t \rangle)$ , where  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$  and  $b(x)$  is uniquely determined by  $c(x)$ .

Case (V) can be proved similarly as that of (IV).  $\square$

Therefore, by Theorems 2.5 and 3.8 we can give all distinct  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$ .

**Corollary 3.9** *Using the notations above, every  $\lambda$ -constacyclic code  $\mathcal{C}$  over  $R$  of length  $p^s n$  can be constructed by the following two steps:*

(i) *For each  $j = 1, \dots, r$ , choose an ideal  $C_j$  of  $\mathcal{K}_j + u\mathcal{K}_j$  listed in Theorem 3.8.*

(ii) *Set  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x)C_j \pmod{x^{p^s n} - \lambda}$ .*

*The number of codewords in  $\mathcal{C}$  is equal to  $|\mathcal{C}| = \prod_{j=1}^r |C_j|$ .*

*Therefore, the number of  $\lambda$ -constacyclic codes over  $R$  of length  $p^s n$  is equal to  $\prod_{j=1}^r N_{(p^m, d_j, p^s)}$ .*

Using the notations of Corollary 3.9(ii),  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x)C_j$  is called the *canonical form decomposition* of the  $\lambda$ -constacyclic code  $\mathcal{C}$ .

As the end of this section, we consider the special situation of  $r = 1$ , i.e.,  $x^n - \lambda_0$  is irreducible in  $\mathbb{F}_{p^m}[x]$ .

**Lemma 3.10** (cf. Wan [18] Theorem 10.7) *Let  $n$  be an integer satisfying  $n \geq 2$ . Let  $\lambda_0 \in \mathbb{F}_{p^m}^\times$  and  $\text{ord}(\lambda_0) = \kappa > 1$  in  $\mathbb{F}_{p^m}^\times$ . Then the binomial  $x^n - \lambda_0 \in \mathbb{F}_{p^m}[x]$  is irreducible over  $\mathbb{F}_{p^m}$  if and only if the following two conditions are satisfied :*

(i) *Every prime divisor of  $n$  divides  $\kappa$ , but not  $(p^m - 1)/\kappa$ .*

(ii) *If  $4|n$  then  $4|(p^m - 1)$ .*

Then by substituting  $\mathcal{K}_j$ ,  $f_j(x)$  and  $d_j$  by  $\mathcal{A}$ ,  $x^n - \lambda_0$  and  $n$  in Theorem 3.8, respectively, we obtain the following corollary.

**Corollary 3.11** *Let  $\lambda_0 \in \mathbb{F}_{p^m}^\times$  satisfying Conditions (i) and (ii) in Lemma 3.10, and denote  $\mathcal{A} = \mathbb{F}_{p^m}[x]/\langle (x^n - \lambda_0)^{p^s} \rangle$ . Then all distinct  $\lambda_0^{p^s}$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$ , i.e., all ideals of the ring  $\mathcal{A} + u\mathcal{A}$ , are given by the following five cases:*

(I)  $p^{(p^s-1-\lceil \frac{1}{2}(p^s-2) \rceil)mn}$  codes:

$$\mathcal{C} = \langle (x^n - \lambda_0)b(x) + u \rangle \text{ with } |\mathcal{C}| = p^{mn p^s},$$

where  $b(x) \in (x^n - \lambda_0)^{\lceil \frac{1}{2}(p^s-2) \rceil} (\mathcal{A}/\langle (x^n - \lambda_0)^{p^s-1} \rangle)$ .

(II)  $\sum_{k=1}^{p^s-1} p^{(p^s-k-1-\lceil \frac{1}{2}(p^s-k-2) \rceil)mn}$  codes:

$$\mathcal{C} = \langle (x^n - \lambda_0)^{k+1}b(x) + u(x^n - \lambda_0)^k \rangle \text{ with } |\mathcal{C}| = p^{mn(p^s-k)},$$

where  $b(x) \in (x^n - \lambda_0)^{\lceil \frac{1}{2}(p^s-k-2) \rceil} (\mathcal{A}/\langle (x^n - \lambda_0)^{p^s-k-1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .

(III)  $p^s + 1$  codes:  $\mathcal{C} = \langle (x^n - \lambda_0)^k \rangle$  with  $|\mathcal{C}| = p^{2mn(p^s-k)}$ ,  $0 \leq k \leq p^s$ .

(IV)  $\sum_{t=1}^{p^s-1} p^{(t-\lceil \frac{t}{2} \rceil)mn}$  codes:

$$\mathcal{C} = \langle (x^n - \lambda_0)b(x) + u, (x^n - \lambda_0)^t \rangle \text{ with } |\mathcal{C}| = p^{mn(2p^s-t)},$$

where  $b(x) \in (x^n - \lambda_0)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{A}/\langle (x^n - \lambda_0)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - 1$ .

(V)  $\sum_{k=1}^{p^s-2} \sum_{t=1}^{p^s-k-1} p^{(t-\lceil \frac{t}{2} \rceil)mn}$  codes:

$$\mathcal{C} = \langle (x^n - \lambda_0)^{k+1}b(x) + u(x^n - \lambda_0)^k, (x^n - \lambda_0)^{k+t} \rangle \text{ with } |\mathcal{C}| = p^{mn(2p^s-2k-t)},$$

where  $b(x) \in (x^n - \lambda_0)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{A}/\langle (x^n - \lambda_0)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

Therefore, the number of  $\lambda_0^{p^s}$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s n$  is equal to

$$N_{(p^m, n, p^s)} = 1 + p^s + \sum_{k=0}^{p^s-1} p^{(p^s-k-1-\lceil \frac{1}{2}(p^s-k-2) \rceil)mn} + \sum_{k=0}^{p^s-2} \sum_{t=1}^{p^s-k-1} p^{(t-\lceil \frac{t}{2} \rceil)mn}.$$

#### 4. Dual codes of $\lambda$ -constacyclic codes over $R$ of length $p^s n$

In this section, we give the dual code of every  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$ .

For any polynomial  $f(x) = \sum_{i=0}^d c_i x^i \in \mathbb{F}_{p^m}[x]$  of degree  $d \geq 1$ , recall that the *reciprocal polynomial* of  $f(x)$  is defined as  $\widetilde{f}(x) = \widetilde{f(x)} = x^d f(\frac{1}{x}) = \sum_{i=0}^d c_i x^{d-i}$ , and  $f(x)$  is said to be *self-reciprocal* if  $\widetilde{f}(x) = \delta f(x)$  for some  $\delta \in \mathbb{F}_{p^m}^\times$ . It is known that  $\widetilde{\widetilde{f}(x)} = f(x)$  if  $f(0) \neq 0$ , and  $\widetilde{f(x)g(x)} = \widetilde{f}(x)\widetilde{g}(x)$  for any monic polynomials  $f(x), g(x) \in \mathbb{F}_{p^m}[x]$  with positive degrees satisfying  $f(0), g(0) \in \mathbb{F}_{p^m}^\times$ .

By  $\lambda_0^{p^s} = \lambda$ , we have  $(\lambda_0^{-1})^{p^s} = \lambda^{-1}$ . Then using the notions and conclusions in Section 2, we have

$$\begin{aligned} x^n - \lambda_0^{-1} &= -\lambda_0^{-1} \widetilde{(x^n - \lambda_0)} = -\lambda_0^{-1} \widetilde{f_1(x)} \dots \widetilde{f_r(x)}, \\ x^{p^s n} - \lambda^{-1} &= -\lambda^{-1} \widetilde{(x^{p^s n} - \lambda)} = -\lambda^{-1} \widetilde{f_1(x)^{p^s}} \dots \widetilde{f_r(x)^{p^s}}, \end{aligned} \quad (6)$$

In the following, we adopt the following notations and definitions.

**Notation 4.1** Let  $\lambda, \lambda_0 \in \mathbb{F}_{p^m}^\times$  satisfying  $\lambda_0^{p^s} = \lambda$ . For any  $1 \leq j \leq r$  we denote

- $\widehat{\mathcal{A}} = \mathbb{F}_{p^m}[x] / \langle x^{p^s n} - \lambda^{-1} \rangle$ ,  $\widehat{\mathcal{A}}[u] / \langle u^2 \rangle = \widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$  ( $u^2 = 0$ );
- $\widehat{\mathcal{K}}_j = \mathbb{F}_{p^m}[x] / \langle \widetilde{f_j(x)^{p^s}} \rangle$ ,  $\widehat{\mathcal{K}}_j[u] / \langle u^2 \rangle = \widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$  ( $u^2 = 0$ );
- $\widehat{\Psi} : \widehat{\mathcal{A}} + u\widehat{\mathcal{A}} \rightarrow R[x] / \langle x^{p^s n} - \lambda^{-1} \rangle$  via

$$\widehat{\Psi} : g_0(x) + u g_1(x) \mapsto \sum_{i=0}^{p^s n - 1} (g_{i,0} + u g_{i,1}) x^i$$

$$(\forall g_k(x) = \sum_{i=0}^{p^s n - 1} g_{i,k} x^i \in \widehat{\mathcal{A}} \text{ with } g_{i,k} \in \mathbb{F}_{p^m}, 0 \leq i \leq p^s n - 1 \text{ and } k = 0, 1).$$

Similar to Lemma 2.1, it can be easily verified that  $\widehat{\Psi}$  is a ring isomorphism from  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$  onto  $\mathcal{R}_{\lambda^{-1}} = R[x] / \langle x^{p^s n} - \lambda^{-1} \rangle$ . Then we will identify  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$  with  $\mathcal{R}_{\lambda^{-1}}$  under  $\widehat{\Psi}$  in the rest of the paper. Now, we define a map

$$\tau : \mathcal{A} \rightarrow \widehat{\mathcal{A}} \text{ via } \tau : a(x) \mapsto a(x^{-1}) \quad (\forall a(x) \in \mathcal{A}).$$

Then one can easily verify that  $\tau$  is a ring isomorphism from  $\mathcal{A}$  onto  $\widehat{\mathcal{A}}$  and can be extended to a ring isomorphism from  $\mathcal{A} + u\mathcal{A}$  onto  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$  in the natural way that  $\tau : \rho(x) \mapsto \rho(x^{-1})$  ( $\forall \rho(x) = \sum_{i=0}^{p^s n - 1} \rho_i x^i \in \mathcal{A} + u\mathcal{A}$  with  $\rho_i \in R$  for all  $i = 0, 1, \dots, p^s n - 1$ ). Moreover, for each  $1 \leq j \leq r$  we define



- $\widehat{\theta}_j(x) = \tau(\theta_j(x)) = \theta_j(x^{-1}) \equiv v_j(x^{-1})F_j(x^{-1}) = 1 - w_j(x^{-1})f_j(x^{-1}) \pmod{x^n - \lambda_0^{-1}};$
- $\widehat{\varepsilon}_j(x) = \tau(\varepsilon_j(x)) = \tau(\theta_j(x)^{p^s}) = \widehat{\theta}_j(x)^{p^s} \pmod{x^{p^s n} - \lambda^{-1}};$
- $\widehat{\Phi} : (\widehat{\mathcal{K}}_1 + u\widehat{\mathcal{K}}_1) \times \dots \times (\widehat{\mathcal{K}}_r + u\widehat{\mathcal{K}}_r) \rightarrow \widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$  via  
 $\widehat{\Phi} : (\xi_1 + u\eta_1, \dots, \xi_r + u\eta_r) \mapsto \sum_{j=1}^r \widehat{\varepsilon}_j(x)(\xi_j + u\eta_j) \pmod{x^{p^s n} - \lambda^{-1}}$   
 $(\forall \xi_j, \eta_j \in \widehat{\mathcal{K}}_j, j = 1, \dots, r);$

•  $\tau_j : \mathcal{K}_j + u\mathcal{K}_j \rightarrow \widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$  via  $\tau_j : \xi \mapsto a(x^{-1}) + ub(x^{-1})$  ( $\forall \xi = a(x) + ub(x) \in \mathcal{K}_j + u\mathcal{K}_j$  with  $a(x), b(x) \in \mathcal{K}_j$ ).

Then similar to Lemma 2.4, we see that  $\widehat{\Phi}$  is a ring isomorphism from  $(\widehat{\mathcal{K}}_1 + u\widehat{\mathcal{K}}_1) \times \dots \times (\widehat{\mathcal{K}}_r + u\widehat{\mathcal{K}}_r)$  onto  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$ . Moreover, we have the following.

**Lemma 4.2** *Using the notations above,  $\tau_j$  is a ring isomorphism from  $\mathcal{K}_j + u\mathcal{K}_j$  onto  $\widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$  satisfying:  $\tau(\varepsilon_j(x)\xi) = \widehat{\varepsilon}_j(x)\tau_j(\xi)$  for all  $\xi \in \mathcal{K}_j + u\mathcal{K}_j$ .*

**Proof.** By (6) we have  $\widetilde{f}_j(x)^{p^s} | (x^{p^s n} - \lambda^{-1})$ , which implies  $x^{p^s n} - \lambda^{-1} = 0$ , i.e.,  $x^{-1} = \lambda x^{p^s n-1}$  in  $\widehat{\mathcal{K}}_j = \mathbb{F}_{p^m}[x]/\langle \widetilde{f}_j(x)^{p^s} \rangle$ . Hence  $\tau_j$  is well defined. By

$$f_j(x^{-1})^{p^s} = \lambda x^{p^s n} f_j(x^{-1})^{p^s} = \lambda x^{p^s(n-d_j)} (x^{d_j} f_j(x^{-1}))^{p^s} = \lambda x^{p^s(n-d_j)} \widetilde{f}_j(x)^{p^s}$$

in  $\widehat{\mathcal{K}}_j$ , one can easily verify that  $\tau_j$  is a ring isomorphism from  $\mathcal{K}_j + u\mathcal{K}_j$  onto  $\widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$ . Finally, for any  $\xi = a(x) + ub(x) \in \mathcal{K}_j + u\mathcal{K}_j$ , where  $a(x), b(x) \in \mathcal{K}_j$ , by the definitions of  $\tau$  and  $\tau_j$  we have that  $\tau(\varepsilon_j(x)\xi) = \varepsilon_j(x^{-1})(a(x^{-1}) + ub(x^{-1})) = \widehat{\varepsilon}_j(x)\tau_j(\xi)$ .  $\square$

**Lemma 4.3** *Let  $a = (a_0, a_1, \dots, a_{p^s n-1}), b = (b_0, b_1, \dots, b_{p^s n-1}) \in R^{p^s n}$  where  $a_i, b_i \in R$  for all  $i = 0, 1, \dots, p^s n - 1$ . We denote*

$$a(x) = \sum_{i=0}^{p^s n-1} a_i x^i \in \mathcal{A} + u\mathcal{A}, \quad b(x) = \sum_{i=0}^{p^s n-1} b_i x^i \in \widehat{\mathcal{A}} + u\widehat{\mathcal{A}}.$$

*Then  $[a, b]_E = \sum_{i=0}^{p^s n-1} a_i b_i = 0$ , if  $\tau(a(x)) \cdot b(x) = 0$  in  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$ .*

**Proof.** By  $x^{p^s} = \lambda^{-1}$  in  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}} = R[x]/\langle x^{p^s n} - \lambda^{-1} \rangle$ , we have  $\lambda x^{p^s} = 1$ , which implies  $\tau(a(x)) = a(x^{-1}) = a_0 + \lambda \sum_{i=1}^{p^s n-1} a_i x^{p^s n-i}$ . Therefore,  $\tau(a(x)) \cdot b(x) = [a, b]_E + \sum_{i=1}^{p^s n-1} c_i x^i$  for some  $c_1, \dots, c_{p^s n-1} \in R$ . Hence  $[a, b]_E = 0$  when  $\tau(a(x)) \cdot b(x) = 0$  in  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$ .  $\square$

Now, we can represent the dual code of each  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$  from its canonical form decomposition.

**Theorem 4.4** *Let  $\mathcal{C}$  be a  $\lambda$ -constacyclic code over  $R$  of length  $p^s n$  with canonical form decomposition  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x) C_j \pmod{x^{p^s n} - \lambda}$ , where  $C_j$  is an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$  listed by Theorem 3.8. Then the dual code  $\mathcal{C}^{\perp_E}$  of  $\mathcal{C}$  is a  $\lambda^{-1}$ -constacyclic code over  $R$  of length  $p^s n$  with canonical form decomposition*

$$\mathcal{C}^{\perp_E} = \bigoplus_{j=1}^r \widehat{\varepsilon}_j(x) \widehat{D}_j \pmod{x^{p^s n} - \lambda^{-1}},$$

where  $\widehat{D}_j$  is an ideal of  $\widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$  given by one of the following five cases:

(I)  $\widehat{D}_j = \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x) b(x^{-1}) + u \rangle$ , if  $C_j = \langle f_j(x) b(x) + u \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - 1} \rangle)$ .

(II)  $\widehat{D}_j = \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x) b(x^{-1}) + u, \widetilde{f}_j(x)^{p^s - k} \rangle$ , if  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - k - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - k - 1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .

(III)  $\widehat{D}_j = \langle \widetilde{f}_j(x)^{p^s - k} \rangle$ , if  $C_j = \langle f_j(x)^k \rangle$  where  $0 \leq k \leq p^s$ .

(IV)  $\widehat{D}_j = \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x)^{p^s - t + 1} b(x^{-1}) + u \widetilde{f}_j(x)^{p^s - t} \rangle$ , if  $C_j = \langle f_j(x) b(x) + u, f_j(x)^t \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(t - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{t - 1} \rangle)$ ,  $1 \leq t \leq p^s - 1$ .

(V)  $\widehat{D}_j = \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x)^{p^s - k - t + 1} b(x^{-1}) + u \widetilde{f}_j(x)^{p^s - k - t}, \widetilde{f}_j(x)^{p^s - k} \rangle$ , if  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k, f_j(x)^{k+t} \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

**Proof.** For each integer  $j$ ,  $1 \leq j \leq r$ , let  $B_j$  be an ideal of  $\mathcal{K}_j + \mathcal{K}_j$  given by one of the following five cases:

(i)  $B_j = \langle -f_j(x) b(x) + u \rangle$ , if  $C_j = \langle f_j(x) b(x) + u \rangle$  is given by Theorem 3.8 (I).

(ii)  $B_j = \langle -f_j(x) b(x) + u, f_j(x)^{p^s - k} \rangle$ , if  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k \rangle$  is given by Theorem 3.8 (II).

(iii)  $B_j = \langle f_j(x)^{p^s - k} \rangle$ , if  $C_j = \langle f_j(x)^k \rangle$  is given by Theorem 3.8 (III).

(iv)  $B_j = \langle -f_j(x)^{p^s - t + 1} b(x) + u f_j(x)^{p^s - t} \rangle$ , if  $C_j = \langle f_j(x) b(x) + u, f_j(x)^t \rangle$  is given by Theorem 3.8 (IV).

(v)  $B_j = \langle -f_j(x)^{p^s-k-t+1}b(x) + uf_j(x)^{p^s-k-t}, f_j(x)^{p^s-k} \rangle$ , if  $C_j = \langle f_j(x)^{k+t}, f_j(x)^{k+1}b(x) + uf_j(x)^k \rangle$  is given by Theorem 3.8 (V).

By Theorem 3.8, one can easily verify that

$$C_j \cdot B_j = \{0\} \text{ and } |C_j||B_j| = p^{2mp^s d_j}. \quad (7)$$

Now, we denote  $\widehat{D}_j = \tau_j(B_j)$ , which is an ideal of  $\widehat{\mathcal{K}}_j + u\widehat{\mathcal{K}}_j$ , and set  $\mathcal{D} = \sum_{j=1}^r \widehat{\varepsilon}_j(x)\widehat{D}_j = \widehat{\Phi}(\widehat{D}_1 \times \dots \times \widehat{D}_r)$ . Then  $\mathcal{D}$  is an ideal of the ring  $\widehat{\mathcal{A}} + u\widehat{\mathcal{A}}$ , i.e., a  $\lambda^{-1}$ -constacyclic code over  $R$  of length  $p^s n$ , and  $\mathcal{D} = \sum_{j=1}^r \tau(\varepsilon_j(x))\tau_j(B_j) = \sum_{j=1}^r \tau(\varepsilon_j(x)B_j) = \tau(\sum_{j=1}^r \varepsilon_j(x)B_j)$  by Lemma 4.2. From this, by Lemma 2.3(i) and (7) we deduce

$$\begin{aligned} \tau(\mathcal{C}) \cdot \mathcal{D} &= \tau\left(\sum_{j=1}^r \varepsilon_j(x)C_j\right) \cdot \mathcal{D} = \tau\left(\left(\sum_{j=1}^r \varepsilon_j(x)C_j\right)\left(\sum_{j=1}^r \varepsilon_j(x)B_j\right)\right) \\ &= \tau\left(\sum_{j=1}^r \varepsilon_j(x)C_j \cdot B_j\right) = \{0\}, \end{aligned}$$

which implies that  $[\xi, \eta]_E = 0$  for all  $\xi \in \mathcal{C}$  and  $\eta \in \mathcal{D}$  by Lemma 4.3. Hence  $\mathcal{D} \subseteq \mathcal{C}^{\perp_E}$ . Moreover, by Corollary 3.9 and (7) it follows that

$$\begin{aligned} |\mathcal{C}||\mathcal{D}| &= \left(\prod_{j=1}^r |C_j|\right) \left(\prod_{j=1}^r |B_j|\right) = \prod_{j=1}^r |C_j||B_j| = p^{2mp^s \sum_{j=1}^r d_j} \\ &= p^{2mp^s n} = |\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}|^{p^s n} = |R|^{p^s n}. \end{aligned}$$

Since  $R$  is a Frobenius ring, we conclude that  $\mathcal{D} = \mathcal{C}^{\perp_E}$  (see [12]).

Finally, we give the explicit representation for each  $\mathcal{C}^{\perp_E}$ . By  $x^{p^s n} - \lambda^{-1} = 0$  in  $\mathcal{R}_{\lambda^{-1}}$ , we have  $\lambda x^{p^s n} = 1$ . From this and by  $\deg(f_j(x)) = d_j < n$ , for any integer  $l$ ,  $1 \leq l \leq p^s - 1$ , we deduce that

$$\tau_j(f_j(x)^l) = f_j(x^{-1})^l = \lambda x^{p^s n - l d_j} (x^{d_j} f_j(x^{-1}))^l = \lambda x^{p^s n - l d_j} \widetilde{f}_j(x)^l. \quad (8)$$

◊ When  $B_j$  is given in Case (i), by (8) we have

$$\begin{aligned} \widehat{D}_j &= \tau_j(B_j) = \langle \tau_j(-f_j(x)b(x) + u) \rangle = \langle -\tau_j(f_j(x))\tau_j(b(x)) + u \rangle \\ &= \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x)b(x^{-1}) + u \rangle \end{aligned}$$

◇ When  $B_j$  is given in Case (ii), by (8) we have

$$\begin{aligned}\widehat{D}_j &= \tau_j(B_j) = \langle \tau_j(-f_j(x)b(x) + u), \tau_j(f_j(x)^{p^s-k}) \rangle \\ &= \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x)b(x^{-1}) + u, \lambda x^{p^s n - (p^s - k)d_j} \widetilde{f}_j(x)^{p^s - k} \rangle \\ &= \langle -\lambda x^{p^s n - d_j} \widetilde{f}_j(x)b(x^{-1}) + u, \widetilde{f}_j(x)^{p^s - k} \rangle.\end{aligned}$$

◇ When  $B_j$  is given in Case (iii), it can be proved similarly as that of Case (ii).

◇ When  $B_j$  is given in Case (iv), by (8) we have

$$\begin{aligned}\widehat{D}_j &= \tau_j(B_j) = \langle \tau_j(-f_j(x)^{p^s-t+1}b(x) + u f_j(x)^{p^s-t}) \rangle \\ &= \langle -\lambda x^{p^s n - (p^s - t + 1)d_j} \widetilde{f}_j(x)^{p^s - t + 1}b(x^{-1}) + u \cdot \lambda x^{p^s n - (p^s - t)d_j} \widetilde{f}_j(x)^{p^s - t} \rangle \\ &= \langle -x^{-d_j} \widetilde{f}_j(x)^{p^s - t + 1}b(x^{-1}) + u \widetilde{f}_j(x)^{p^s - t} \rangle,\end{aligned}$$

where  $x^{-d_j} = \lambda x^{p^s n - d_j}$  in  $\widehat{\mathcal{K}}_j = \mathbb{F}_{p^m}[x]/\langle \widetilde{f}_j(x)^{p^s} \rangle$  as  $\widetilde{f}_j(x)^{p^s} | (x^{p^s n} - \lambda^{-1})$ .

◇ When  $B_j$  is given in Case (v), it can be proved similarly as that of Cases (ii) and (iv).  $\square$

**Example 4.5** By [18] Example 10.1,  $x^{2^i} - 3$  is irreducible over  $\mathbb{F}_5$  for any integer  $i \geq 0$ . It is clear that  $3^5 = 3$  and  $x^{2^i} - 3 = x^{2^i} + 2$  in  $\mathbb{F}_5[x]$ . Now, we consider the special case of  $i = 2$ . By  $p = 5$ ,  $s = 1$  and  $n = 4$ , we have

$$\begin{aligned}\diamond \mathcal{T} &= \{\sum_{i=0}^3 a_i x^i \mid a_0, a_1, a_2, a_3 \in \mathbb{F}_5\} \text{ and } |\mathcal{T}| = 5^4; \\ \diamond \mathcal{A} &= \mathbb{F}_5[x]/\langle (x^4 + 2)^5 \rangle = \{\sum_{j=0}^4 b_j(x)(x^4 + 2)^j \mid b_j(x) \in \mathcal{T}, j = 0, 1, 2, 3, 4\}; \\ \diamond (x^4 + 2)^l(\mathcal{A}/\langle (x^4 + 2)^t \rangle) &= \{\sum_{j=l}^{t-1} b_j(x)(x^4 + 2)^j \mid b_j(x) \in \mathcal{T}, j = l, \dots, t-1\} \\ \text{for any } 0 \leq l \leq t \leq 4, &\text{ where we set } (x^4 + 2)^t(\mathcal{A}/\langle (x^4 + 2)^t \rangle) = \{0\}.\end{aligned}$$

As  $\lambda = 3$ ,  $\widetilde{(x^4 + 2)} = 1 + 2x^4 = 2(x^4 + 3)$  and  $2 \cdot (-\lambda) = 4$ , by Corollary 3.11 and Theorem 4.4 we see that all distinct 3-constacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 20 and their dual codes are given by one the following five cases:

(I)  $5^{4 \cdot 2} = 390625$  codes:

$$\mathcal{C} = \langle (x^4 + 2)b(x) + u \rangle \text{ with } |\mathcal{C}| = 5^{20}$$

and  $\mathcal{C}^{\perp_E} = \langle 4x^{16}(x^4 + 3)b(x^{-1}) + u \rangle$  which is a 2-constacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 20, where  $b(x) \in (x^4 + 2)^2(\mathcal{A}/\langle (x^4 + 2)^4 \rangle) = \{g(x)(x^4 + 2)^2 + h(x)(x^4 + 2)^3 \mid g(x), h(x) \in \mathcal{T}\}$ .

(II)  $5^{4 \cdot 2} + 5^{4 \cdot 1} + 5^{4 \cdot 1} + 5^{4 \cdot 0} = 391876$  codes:

$$\mathcal{C} = \langle (x^4 + 2)^{k+1}b(x) + u(x^4 + 2)^k \rangle \text{ with } |\mathcal{C}| = 5^{4(5-k)}$$

and  $\mathcal{C}^{\perp_E} = \langle 4x^{16}(x^4 + 3)b(x^{-1}) + u, (x^4 + 3)^{5-k} \rangle$ , where

$$b(x) \in (x^4 + 2)^{\lceil \frac{1}{2}(5-k-2) \rceil} (\mathcal{A} / \langle (x^4 + 2)^{4-k} \rangle) \text{ and } 1 \leq k \leq 4.$$

(III) 6 codes:  $\mathcal{C} = \langle (x^4 + 2)^k \rangle$  with  $|\mathcal{C}| = 5^{8(5-k)}$  and  $\mathcal{C}^{\perp_E} = \langle (x^4 + 3)^{5-k} \rangle$ , where  $0 \leq k \leq 5$ .

(IV)  $5^{4 \cdot 0} + 5^{4 \cdot 1} + 5^{4 \cdot 1} + 5^{4 \cdot 2} = 391876$  codes:

$$\mathcal{C} = \langle (x^4 + 2)b(x) + u, (x^4 + 2)^t \rangle \text{ with } |\mathcal{C}| = 5^{4(10-t)}$$

and  $\mathcal{C}^{\perp_E} = \langle 4x^{16}(x^4 + 3)^{5-t+1}b(x^{-1}) + u(x^4 + 3)^{5-t} \rangle$ , where  $b(x) \in (x^4 + 2)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{A} / \langle (x^4 + 2)^{t-1} \rangle)$ ,  $1 \leq t \leq 4$ .

(V)  $(5^{4 \cdot 0} + 5^{4 \cdot 1} + 5^{4 \cdot 1}) + (5^{4 \cdot 0} + 5^{4 \cdot 1}) + 5^{4 \cdot 0} = 1878$  codes:

$$\mathcal{C} = \langle (x^4 + 2)^{k+1}b(x) + u(x^4 + 2)^k, (x^4 + 2)^{k+t} \rangle \text{ with } |\mathcal{C}| = 5^{4(10-2k-t)}$$

and  $\mathcal{C}^{\perp_E} = \langle 4x^{16}(x^4 + 3)^{5-k-t+1}b(x^{-1}) + u(x^4 + 3)^{p^s-k-t}, (x^4 + 3)^{p^s-k} \rangle$ , where  $b(x) \in (x^4 + 2)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{A} / \langle (x^4 + 2)^{t-1} \rangle)$ ,  $1 \leq t \leq 4 - k$  and  $1 \leq k \leq 3$ .

Therefore, the number of 3-constacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 20 is equal to 1176261, and the only self-dual 3-constacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 20 is  $\langle u \rangle = u\mathbb{F}_5^{20}$  (corresponding to  $b(x) = 0$  in Case (I)).

## 5. Self-dual negacyclic codes of length $p^s n$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

In this section, we investigate self-dual negacyclic codes of length  $p^s n$  over  $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ . Let  $\lambda = -1$ . Using the notations in Sections 1–4, by  $\lambda^{-1} = \lambda = -1$  we have  $\mathcal{R}_\lambda = \mathcal{R}_{\lambda^{-1}} = \mathcal{R}_{-1}$  and  $\mathcal{A} = \widehat{\mathcal{A}} = \mathbb{F}_{p^m}[x] / \langle x^{p^s n} + 1 \rangle$ . Hence the map  $\tau : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $\tau(a(x)) = a(x^{-1})$  ( $\forall a(x) \in \mathcal{A}$ ) is a ring automorphism on  $\mathcal{A}$  satisfying  $\tau^{-1} = \tau$ .

By Equations (1) in Section 2 and (6) in Section 4, we have

$$x^{p^s n} + 1 = f_1(x)^{p^s} \dots f_r(x)^{p^s} \text{ and } x^{p^s n} + 1 = \widetilde{f}_1(x)^{p^s} \dots \widetilde{f}_r(x)^{p^s}.$$

Since  $f_1(x), \dots, f_r(x)$  are pairwise coprime monic irreducible polynomials in  $\mathbb{F}_{p^m}[x]$ , for each  $1 \leq j \leq r$  there is a unique integer  $j'$ ,  $1 \leq j' \leq r$ , such that

$\widetilde{f}_j(x) = \delta_j f_{j'}(x)$  where  $\delta_j = f_j(0)^{-1} \in \mathbb{F}_{p^m}^\times$ . In the following, we still denote the bijection  $j \mapsto j'$  on the set  $\{1, \dots, r\}$  by  $\tau$ , i.e.,

$$\widetilde{f}_j(x) = \delta_j f_{\tau(j)}(x).$$

Whether  $\tau$  denotes the automorphism of  $\mathcal{A}$  or this map on the set  $\{1, 2, \dots, r\}$  is determined by context. The next lemma shows the compatibility of the two uses of  $\tau$ .

**Lemma 5.1** *With the notations above, the following hold.*

- (i)  $\tau$  is a permutation on  $\{1, \dots, r\}$  satisfying  $\tau^{-1} = \tau$ .
- (ii) After a suitable rearrangement of  $f_1(x), \dots, f_r(x)$ , there are non-negative integers  $\rho, \epsilon$  such that  $\rho + 2\epsilon = r$ ,  $\tau(j) = j$  for all  $j = 1, \dots, \rho$ ,  $\tau(\rho + i) = \rho + \epsilon + i$  and  $\tau(\rho + \epsilon + i) = \rho + i$  for all  $i = 1, \dots, \epsilon$ .
- (iii) For any  $1 \leq j \leq r$ ,  $\tau(\varepsilon_j(x)) = \widehat{\varepsilon}_j(x) = \varepsilon_{\tau(j)}(x)$  in the ring  $\mathcal{A}$ .
- (iv) For any integer  $j$ ,  $1 \leq j \leq r$ , the map  $\tau_j : \mathcal{K}_j + u\mathcal{K}_j \rightarrow \mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)}$  defined by  $\tau_j(a(x) + ub(x)) = a(x^{-1}) + ub(x^{-1})$  ( $\forall a(x), b(x) \in \mathcal{K}_j$ ) is a ring isomorphism from  $\mathcal{K}_j + u\mathcal{K}_j$  onto  $\mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)}$  with inverse  $\tau_j^{-1} = \tau_{\tau(j)}$ .  
Moreover, we have  $\tau(\varepsilon_j(x)\xi) = \varepsilon_{\tau(j)}(x)\tau_j(\xi)$  for all  $\xi \in \mathcal{K}_j + u\mathcal{K}_j$ .

**Proof.** (i) By  $\widetilde{f}_j(x) = \delta_j f_{\tau(j)}(x)$  for all  $j$ , it follows that  $f_{\tau(j)}(x) = \delta_j^{-1} \widetilde{f}_j(x)$ , which implies  $\widetilde{f_{\tau(j)}}(x) = \delta_j^{-1} \widetilde{\widetilde{f}_j}(x) = \delta_j^{-1} f_j(x)$ , and so  $\delta_{\tau(j)} = \delta_j^{-1}$ . Hence  $f_{\tau^2(j)}(x) = f_{\tau(\tau(j))}(x) = \delta_{\tau(j)}^{-1} \widetilde{f_{\tau(j)}}(x) = \delta_{\tau(j)}^{-1} \delta_j^{-1} \widetilde{f_j}(x) = f_j(x)$ , which implies  $\tau^2(j) = j$  for all  $j \in \{1, \dots, r\}$ . Therefore  $\tau^{-1} = \tau$ .

(ii) It follows from (i) immediately.

(iii) By the equation  $v_j(x)F_j(x) + w_j(x)f_j(x) = 1$  in Section 2, we deduce that  $v_j(x^{-1})F_j(x^{-1}) + w_j(x^{-1})f_j(x^{-1}) = \tau(v_j(x)F_j(x) + w_j(x)f_j(x)) = 1$  in  $\mathcal{A}$ , where  $f_j(x^{-1}) = x^{-d_j} \widetilde{f_j}(x) = \delta_j x^{-d_j} f_{\tau(j)}(x)$  and

$$F_j(x^{-1}) = \frac{x^{-n} + 1}{f_j(x^{-1})} = \frac{x^n + 1}{\delta_j x^{n-d_j} f_{\tau(j)}(x)} = \delta_j^{-1} x^{d_j-n} F_{\tau(j)}(x).$$

Hence  $(v_j(x^{-1})\delta_j^{-1} x^{d_j-n})F_{\tau(j)}(x) + (w_j(x^{-1})\delta_j x^{-d_j})f_{\tau(j)}(x) = 1$  in  $\mathcal{A}$ . From these and by the definition of  $\widehat{\theta}_j(x)$ , we deduce that

$$\widehat{\theta}_j(x) \equiv (v_j(x^{-1})\delta_j^{-1} x^{d_j-n})F_{\tau(j)}(x) = 1 - (w_j(x^{-1})\delta_j x^{-d_j})f_{\tau(j)}(x) \pmod{x^n + 1},$$

which implies  $\widehat{\theta}_j(x) = \theta_{\tau(j)}(x)$  by Notation 2.2. Then by Notation 2.2 and the definition of  $\widehat{\varepsilon}_j(x)$ , it follows that

$$\tau(\varepsilon_j(x)) = \widehat{\varepsilon}_j(x) = \widehat{\theta}_j(x)^{p^s} = \theta_{\tau(j)}(x)^{p^s} = \varepsilon_{\tau(j)}(x), \quad \forall j = 1, \dots, r.$$

(iv) By Notation 4.1 and  $\widetilde{f}_j(x) = \delta_j f_{\tau(j)}(x)$ , we have

$$\widehat{\mathcal{K}}_j = \mathbb{F}_{p^m}[x]/\langle \widetilde{f}_j(x)^{p^s} \rangle = \mathbb{F}_{p^m}[x]/\langle \delta_j^{p^s} f_{\tau(j)}(x)^{p^s} \rangle = \mathbb{F}_{p^m}[x]/\langle f_{\tau(j)}(x)^{p^s} \rangle = \mathcal{K}_{\tau(j)}.$$

Then by Lemma 4.2 we deduce that  $\tau_j$  is a ring isomorphism from  $\mathcal{K}_j + u\mathcal{K}_j$  onto  $\mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)}$  and  $\tau(\varepsilon_j(x)\xi) = \varepsilon_{\tau(j)}(x)\tau_j(\xi)$  for all  $\xi \in \mathcal{K}_j + u\mathcal{K}_j$ .  $\square$

Then by  $\lambda = -1$ , Lemma 5.1 and Theorem 4.4, we have the following conclusion for dual codes of negacyclic codes over  $R$  of length  $p^s n$ .

**Corollary 5.2** *Let  $\mathcal{C}$  be a negacyclic code over  $R$  of length  $p^s n$  with canonical form decomposition  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x)C_j \pmod{x^{p^s n} + 1}$ , where  $C_j$  is an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$  listed by Theorem 3.8. Then the dual code  $\mathcal{C}^{\perp_E}$  of  $\mathcal{C}$  is a negacyclic code over  $R$  of length  $p^s n$  with canonical form decomposition*

$$\mathcal{C}^{\perp_E} = \bigoplus_{j=1}^r \varepsilon_{\tau(j)}(x)D_{\tau(j)} \pmod{x^{p^s n} + 1},$$

where  $D_{\tau(j)}$  is an ideal of  $\mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)}$  given by one of the following five cases:

(I) If  $C_j = \langle f_j(x)b(x) + u \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - 2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{p^s - 1} \rangle)$ ,

$$D_{\tau(j)} = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)b(x^{-1}) + u \rangle.$$

(II) If  $C_j = \langle f_j(x)^{k+1}b(x) + u f_j(x)^k \rangle$  where  $1 \leq k \leq p^s - 1$  and  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - k - 2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{p^s - k - 1} \rangle)$ ,

$$D_{\tau(j)} = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)b(x^{-1}) + u, f_{\tau(j)}(x)^{p^s - k} \rangle.$$

(III) If  $C_j = \langle f_j(x)^k \rangle$  where  $0 \leq k \leq p^s$ ,  $D_{\tau(j)} = \langle f_{\tau(j)}(x)^{p^s - k} \rangle$ .

(IV) If  $C_j = \langle f_j(x)b(x) + u, f_j(x)^t \rangle$  where  $1 \leq t \leq p^s - 1$  and  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(t - 2) \rceil}(\mathcal{K}_j / \langle f_j(x)^{t - 1} \rangle)$ ,

$$D_{\tau(j)} = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)^{p^s - t + 1} b(x^{-1}) + u f_{\tau(j)}(x)^{p^s - t} \rangle.$$

(V) If  $C_j = \langle f_j(x)^{k+1}b(x) + u f_j(x)^k, f_j(x)^{k+t} \rangle$  where  $1 \leq t \leq p^s - k - 1$ ,  $1 \leq k \leq p^s - 2$ , and  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$ ,

$$D_{\mu(j)} = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)^{p^s - k - t + 1} b(x^{-1}) + u f_{\tau(j)}(x)^{p^s - k - t}, f_{\tau(j)}(x)^{p^s - k} \rangle.$$

Now, we can determine self-dual negacyclic codes of length  $p^s n$  over  $R$  by the following theorem.

**Theorem 5.3** *Using the notations in Lemma 5.1(ii), all distinct self-dual negacyclic codes of length  $p^s n$  over  $R$  are given by  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_j(x) C_j \pmod{x^{p^s n} + 1}$ , where  $C_j$  is an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$  given by one of the following two cases:*

(i) *If  $1 \leq j \leq \rho$ ,  $C_j$  is given by one of the following three subcases.*

(i-1)  $C_j = \langle f_j(x)b(x) + u \rangle$ , where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - 1} \rangle)$  satisfying  $b(x) - \delta_j x^{p^s n - d_j} b(x^{-1}) \equiv 0 \pmod{f_j(x)^{p^s - 1}}$ .

(i-2)  $C_j = \langle f_j(x)^k \rangle$  where  $k$  is an integer satisfying  $2k = p^s$ .

(i-3)  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k, f_j(x)^{k+t} \rangle$ , where  $1 \leq t \leq p^s - k - 1$ ,  $1 \leq k \leq p^s - 2$ , and  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$  satisfying  $p^s = 2k + t$  and  $b(x) - \delta_j x^{p^s n - d_j} b(x^{-1}) \equiv 0 \pmod{f_j(x)^{t-1}}$ .

(ii) *If  $j = \rho + i$  where  $1 \leq i \leq \epsilon$ , the pair  $(C_j, C_{j+\epsilon})$  of ideals is given by one of the following five subcases.*

(ii-1)  $C_j = \langle f_j(x)b(x) + u \rangle$  and  $C_{j+\epsilon} = \langle \delta_j x^{p^s n - d_j} f_{j+\epsilon}(x)b(x^{-1}) + u \rangle$  where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - 1} \rangle)$ .

(ii-2)  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k \rangle$  and

$$C_{j+\epsilon} = \langle \delta_j x^{p^s n - d_j} f_{j+\epsilon}(x)b(x^{-1}) + u, f_{j+\epsilon}(x)^{p^s - k} \rangle,$$

where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - k - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - k - 1} \rangle)$  and  $1 \leq k \leq p^s - 1$ .

(ii-3)  $C_j = \langle f_j(x)^k \rangle$  and  $C_{j+\epsilon} = \langle f_{j+\epsilon}(x)^{p^s - k} \rangle$ , where  $0 \leq k \leq p^s$ .

(ii-4)  $C_j = \langle f_j(x)b(x) + u, f_j(x)^t \rangle$  and

$$C_{j+\epsilon} = \langle \delta_j x^{p^s n - d_j} f_{j+\epsilon}(x)^{p^s - t + 1} b(x^{-1}) + u f_{j+\epsilon}(x)^{p^s - t} \rangle,$$

where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(t-2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$  and  $1 \leq t \leq p^s - 1$ .

(ii-5)  $C_j = \langle f_j(x)^{k+1} b(x) + u f_j(x)^k, f_j(x)^{k+t} \rangle$  and

$$C_{j+\epsilon} = \langle \delta_j x^{p^s n - d_j} f_{j+\epsilon}(x)^{p^s - k - t + 1} b(x^{-1}) + u f_{j+\epsilon}(x)^{p^s - k - t}, f_{j+\epsilon}(x)^{p^s - k} \rangle,$$



where  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$ ,  $1 \leq t \leq p^s - k - 1$  and  $1 \leq k \leq p^s - 2$ .

**Proof.** By Corollary 5.2, we have  $\mathcal{C}^{\perp_E} = \bigoplus_{j=1}^r \varepsilon_{\tau(j)}(x) D_{\tau(j)}$ , where  $D_{\tau(j)}$  is an ideal of  $\mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)}$  given by Corollary 5.2 for  $j = 1, \dots, r$ . Since  $\tau$  is a bijection on the set  $\{1, \dots, r\}$ , we have  $\mathcal{C} = \bigoplus_{j=1}^r \varepsilon_{\tau(j)}(x) C_{\tau(j)}$ . From this and by Theorem 2.5(iii), we deduce that  $\mathcal{C}$  is self-dual if and only if  $C_{\tau(j)} = D_{\tau(j)}$  for all  $j = 1, \dots, r$ . Then by Lemma 5.1(ii), we have one of the following two cases.

(i) Let  $1 \leq j \leq \rho$ . Then  $\tau(j) = j$ . By Corollary 5.2,  $C_j$  satisfies the condition  $C_{\tau(j)} = D_{\tau(j)}$  if and only if  $C_j$  is given one of the following subcases:

(i-1)  $C_j = \langle f_j(x)b(x) + u \rangle = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)b(x^{-1}) + u \rangle$ , where  $b(x) \in f_j(x)^{\lceil \frac{1}{2}(p^s - 2) \rceil} (\mathcal{K}_j / \langle f_j(x)^{p^s - 1} \rangle)$  satisfying

$$b(x) - \delta_j x^{p^s n - d_j} b(x^{-1}) \equiv 0 \pmod{f_j(x)^{p^s - 1}}.$$

(i-2)  $C_j = \langle f_j(x)^k \rangle = \langle f_{\tau(j)}(x)^{p^s - k} \rangle$ , where  $0 \leq k \leq p^s$  satisfying  $p^s - k = k$ , i.e.,  $2k = p^s$ .

(i-3)  $C_j = \langle f_j(x)^{k+1}b(x) + u f_j(x)^k, f_j(x)^{k+t} \rangle$  and

$$C_j = \langle \delta_j x^{p^s n - d_j} f_{\tau(j)}(x)^{p^s - k - t + 1} b(x^{-1}) + u f_{\tau(j)}(x)^{p^s - k - t}, f_{\tau(j)}(x)^{p^s - k} \rangle,$$

where  $1 \leq t \leq p^s - k - 1$ ,  $1 \leq k \leq p^s - 2$ , and  $b(x) \in f_j(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_j / \langle f_j(x)^{t-1} \rangle)$  satisfying  $k + t = p^s - k$ , i.e.,  $p^s = 2k + t$  and

$$b(x) - \delta_j x^{p^s n - d_j} b(x^{-1}) \equiv 0 \pmod{f_j(x)^{t-1}}.$$

(ii) Let  $j = \rho + i$  where  $1 \leq i \leq \epsilon$ . Then  $\tau(j) = j + \epsilon$ . In this case, we choose an arbitrary ideal  $C_j$  of  $\mathcal{K}_j + u\mathcal{K}_j$  listed in Theorem 3.8 and let  $C_{j+\epsilon} = C_{\tau(j)} = D_{\tau(j)} = \tau_j(B_j)$ , where  $D_{\tau(j)}$  is given by Corollary 5.2 and  $B_j$  is given by the proof of Theorem 4.4 with  $\tau(j) = j + \epsilon$ , respectively. Then the condition  $C_{\tau(j)} = D_{\tau(j)}$  is satisfied for all  $j = \rho + 1, \dots, \rho + \epsilon$ .

Moreover, by Equation (7) in the proof of Theorem 4.4 we have  $C_j \cdot B_j = \{0\}$  and  $|C_j||B_j| = p^{2mp^s d_j}$ , where  $B_j$  is an ideal of  $\mathcal{K}_j + u\mathcal{K}_j$ . By Lemma 5.1(ii) and (iv), we know that  $\tau_j$  is a ring isomorphism from  $\mathcal{K}_j + u\mathcal{K}_j$  onto  $\mathcal{K}_{\tau(j)} + u\mathcal{K}_{\tau(j)} = \mathcal{K}_{j+\epsilon} + u\mathcal{K}_{j+\epsilon}$  with inverse  $\tau_j^{-1} = \tau_{\tau(j)} = \tau_{j+\epsilon}$ , which implies that  $\tau_j(B_j)$  and  $\tau_j(C_j)$  are ideals of  $\mathcal{K}_{j+\epsilon} + u\mathcal{K}_{j+\epsilon}$  satisfying

$$\tau_j(B_j) \cdot \tau_j(C_j) = \{0\} \text{ and } |\tau_j(B_j)||\tau_j(C_j)| = |B_j||C_j| = p^{2mp^s d_j} = p^{2mp^s d_{j+\epsilon}}.$$

From this, by  $C_{j+\epsilon} = \tau_j(B_j)$  and the proof of Theorem 4.4 we deduce  $B_{j+\epsilon} = \tau_j(C_j)$ , which implies  $D_{\tau(j+\epsilon)} = \tau_{j+\epsilon}(B_{j+\epsilon}) = \tau_{j+\epsilon}(\tau_j(C_j)) = C_j = C_{\tau(j+\epsilon)}$ , for all  $j = \rho + 1, \dots, \rho + \epsilon$ .

As stated above, we conclude that the condition  $C_{\tau(j)} = D_{\tau(j)}$  is satisfied for all  $j = \rho + 1, \dots, \rho + \epsilon, \rho + \epsilon + 1, \dots, \rho + 2\epsilon$ .  $\square$

## 6. Negacyclic codes over $\mathbb{F}_5 + u\mathbb{F}_5$ of length $2 \cdot 5^s \cdot 3^t$

In this section, let  $t$  be a positive integer. We consider negacyclic codes of length  $2 \cdot 5^s \cdot 3^t$  over  $\mathbb{F}_5 + u\mathbb{F}_5$ . In this case, we have  $p = 5$ ,  $m = 1$ ,  $n = 2 \cdot 3^t$  and  $\lambda = -1$ . By [3] Section 4, we know that

$$x^{2 \cdot 3^t} + 1 = \prod_{i=1}^{t+1} f_i(x) f_{t+1+i}(x)$$

is the factorization of  $x^{2 \cdot 3^t} + 1$  into monic irreducible factors in  $\mathbb{F}_5[x]$ , where

$$\begin{aligned} f_1(x) &= x + 2 \text{ with degree } d_1 = \deg(f_1(x)) = 1, f_{t+2}(x) = 3\tilde{f}_1(x) = x + 3; \\ f_i(x) &= x^{2 \cdot 3^{i-2}} + 2x^{3^{i-2}} + 4 \text{ with degree } d_i = \deg(f_i(x)) = 2 \cdot 3^{i-2}, \\ f_{t+1+i}(x) &= 4\tilde{f}_i(x) = x^{2 \cdot 3^{i-2}} + 3x^{3^{i-2}} + 4, \text{ for } i = 2, \dots, t+1. \end{aligned}$$

Hence  $\rho = 0$ ,  $\epsilon = t + 1$ ,  $\tilde{f}_i(x) = \delta_i f_{t+1+i}(x)$  where  $\delta_1 = 3$  and  $\delta_i = 4$  for all  $i = 2, \dots, t + 1$ . Then

$$x^{2 \cdot 5^s \cdot 3^t} + 1 = (x^{2 \cdot 3^t} + 1)^{5^s} = \prod_{i=1}^{t+1} f_i(x)^{5^s} f_{t+1+i}(x)^{5^s}.$$

For any integer  $i$ ,  $1 \leq i \leq t + 1$ , we find polynomials  $a_i(x), b_i(x) \in \mathbb{F}_5[x]$  satisfying  $a_i(x) \frac{x^{2 \cdot 3^t} + 1}{f_i(x)} + b_i(x) f_i(x) = 1$ . Then set

$$\theta_i(x) \equiv a_i(x) \frac{x^{2 \cdot 3^t} + 1}{f_i(x)} = 1 - b_i(x) f_i(x) \pmod{x^{2 \cdot 3^t} + 1},$$

$$\varepsilon_i(x) = \theta_i(x)^{5^s} = \theta_i(x^{5^s}) \text{ and } \varepsilon_{t+1+i}(x) = \varepsilon_i(x^{-1}).$$

Denote  $\mathcal{K}_i = \mathbb{F}_5[x] / \langle f_i(x)^{5^s} \rangle$  and  $\mathcal{T}_i = \{ \sum_{j=0}^{d_i-1} a_j x^j \mid a_0, a_1, \dots, a_{d_i-1} \in \mathbb{F}_5 \}$  for all  $i = 1, 2, \dots, 2(t + 1)$ . By Theorem 3.8, Corollary 3.9 and Lemma 5.1, all distinct negacyclic code over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length  $2 \cdot 5^s \cdot 3^t$  are given by:

$$\mathcal{C} = \bigoplus_{i=1}^{2(t+1)} \varepsilon_i(x) C_i, \quad (9)$$

where  $C_i$  is an ideal of the ring  $\mathcal{K}_i + u\mathcal{K}_i$  ( $u^2 = 0$ ),  $1 \leq i \leq 2(t+1)$ , given by one of the following five cases:

(I-i)  $5^{(5^s-1-\lceil \frac{1}{2}(5^s-2) \rceil)d_i}$  ideals:

$$C_i = \langle f_i(x)b(x) + u \rangle \text{ with } |C_i| = p^{5^s d_i},$$

where  $b(x) \in f_i(x)^{\lceil \frac{1}{2}(5^s-2) \rceil} (\mathcal{K}_i / \langle f_i(x)^{5^s-1} \rangle)$ .

(II-i)  $\sum_{k=1}^{5^s-1} 5^{(5^s-k-1-\lceil \frac{1}{2}(5^s-k-2) \rceil)d_i}$  ideals:

$$C_i = \langle f_i(x)^{k+1}b(x) + u f_i(x)^k \rangle \text{ with } |C_i| = 5^{(5^s-k)d_i},$$

where  $b(x) \in f_i(x)^{\lceil \frac{1}{2}(5^s-k-2) \rceil} (\mathcal{K}_i / \langle f_i(x)^{5^s-k-1} \rangle)$  and  $1 \leq k \leq 5^s - 1$ .

(III-i)  $5^s + 1$  ideals:  $C_i = \langle f_i(x)^k \rangle$  with  $|C_i| = p^{2(5^s-k)d_i}$ ,  $0 \leq k \leq 5^s$ .

(IV-i)  $\sum_{t=1}^{5^s-1} 5^{(t-\lceil \frac{t}{2} \rceil)d_i}$  ideals:

$$C_i = \langle f_i(x)b(x) + u, f_i(x)^t \rangle \text{ with } |C_i| = 5^{(2 \cdot 5^s - t)d_i},$$

where  $b(x) \in f_i(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_i / \langle f_i(x)^{t-1} \rangle)$ ,  $1 \leq t \leq 5^s - 1$ .

(V-i)  $\sum_{k=1}^{5^s-2} \sum_{t=1}^{5^s-k-1} 5^{(t-\lceil \frac{t}{2} \rceil)d_i}$  ideals:

$$C_i = \langle f_i(x)^{k+1}b(x) + u f_i(x)^k, f_i(x)^{k+t} \rangle \text{ with } |C_i| = 5^{(2 \cdot 5^s - 2k - t)d_i},$$

where  $b(x) \in f_i(x)^{\lceil \frac{t-2}{2} \rceil} (\mathcal{K}_i / \langle f_i(x)^{t-1} \rangle)$ ,  $1 \leq t \leq 5^s - k - 1$  and  $1 \leq k \leq 5^s - 2$ .

Moreover, the number of codewords contained in the code  $\mathcal{C}$  given by (9) is equal to  $\prod_{i=1}^{t+1} |C_i|$ , and the numbers of all negacyclic codes of length  $2 \cdot 5^s \cdot 3^t$  over  $\mathbb{F}_5 + u\mathbb{F}_5$  is equal to  $\prod_{i=1}^{t+1} N_{(5, d_i, 5^s)}^2$  where

$$N_{(5, d_i, 5^s)} = 1 + 5^s + \sum_{k=0}^{5^s-1} 5^{(5^s-k-1-\lceil \frac{1}{2}(5^s-k-2) \rceil)d_i} + \sum_{k=0}^{5^s-2} \sum_{t=1}^{5^s-k-1} p^{(t-\lceil \frac{t}{2} \rceil)d_i}$$

for all  $i = 1, 2, \dots, t+1$ .

By Theorem 5.3, the number of self-dual negacyclic codes of length  $2 \cdot 5^s \cdot 3^t$  over  $\mathbb{F}_5 + u\mathbb{F}_5$  is equal to  $\prod_{i=1}^{t+1} N_{(5, d_i, 5^s)}$  and all distinct self-dual negacyclic codes of length  $2 \cdot 5^s \cdot 3^t$  over  $\mathbb{F}_5 + u\mathbb{F}_5$  are given by the following:

$$\mathcal{C} = \bigoplus_{i=1}^{t+1} (\varepsilon_i(x)C_i \oplus \varepsilon_{t+1+i}(x)C_{t+1+i}),$$

where  $C_i$  is an ideal of the ring  $\mathcal{K}_i + u\mathcal{K}_i$  ( $u^2 = 0$ ) given by (I-i)–(V-i) above and  $C_{t+1+i}$  is an ideal of the ring  $\mathcal{K}_{t+1+i} + u\mathcal{K}_{t+1+i}$  ( $u^2 = 0$ ),  $1 \leq i \leq t$ , given by the one of the following five cases:

(i-1) If  $C_i$  is given in Case (I-i),  $C_{t+1+i} = \langle \delta_i x^{2 \cdot 5^s \cdot 3^t - d_i} f_{t+1+i}(x) b(x^{-1}) + u \rangle$ .

(i-2) If  $C_i$  is given in Case (II-i),  $C_{t+1+i} = \langle \delta_i x^{2 \cdot 5^s \cdot 3^t - d_i} f_{t+1+i}(x) b(x^{-1}) + u, f_{t+1+i}(x)^{5^s - k} \rangle$ .

(i-3) If  $C_i$  is given in Case (III-i),  $C_{t+1+i} = \langle f_{t+1+i}(x)^{5^s - k} \rangle$ .

(i-4) If  $C_i$  is given in Case (IV-i),

$$C_{t+1+i} = \langle \delta_i x^{2 \cdot 5^s \cdot 3^t - d_i} f_{t+1+i}(x)^{5^s - t + 1} b(x^{-1}) + u f_{t+1+i}(x)^{5^s - t} \rangle.$$

(i-5) If  $C_i$  is given in Case (V-i),

$$C_{t+1+i} = \langle \delta_i x^{2 \cdot 5^s \cdot 3^t - d_i} f_{t+1+i}(x)^{5^s - k - t + 1} b(x^{-1}) + u f_{t+1+i}(x)^{5^s - k - t}, f_{t+1+i}(x)^{5^s - k} \rangle.$$

Finally, we consider negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 30 corresponding to the special case of  $s = t = 1$ . In this case,  $x^6 + 1 = f_1(x)f_2(x)f_3(x)f_4(x)$  is the factorization of  $x^6 + 1$  into monic irreducible factors in  $\mathbb{F}_5[x]$ , where  $f_1(x) = x + 2$ ,  $f_2(x) = x^2 + 2x + 4$ ,  $f_3(x) = x + 3$  and  $f_4(x) = x^2 + 3x + 4$  satisfying  $\tilde{f}_1(x) = \delta_1 f_3(x)$  and  $\tilde{f}_2(x) = \delta_1 f_4(x)$  with  $\delta_1 = 3$  and  $\delta_2 = 4$ . First, we obtain the following:

$$\varepsilon_1(x) = 2x^{25} + x^{20} + 3x^{15} + 4x^{10} + 2x^5 + 1;$$

$$\varepsilon_2(x) = 2x^{25} + 4x^{20} + 4x^{15} + x^{10} + 2x^5 + 2;$$

$$\varepsilon_3(x) = 3x^{25} + x^{20} + 2x^{15} + 4x^{10} + 3x^5 + 1;$$

$$\varepsilon_4(x) = 3x^{25} + 4x^{20} + x^{15} + x^{10} + 3x^5 + 2.$$

Next, let  $\mathcal{K}_j = \mathbb{F}_5[x]/\langle f_j(x)^5 \rangle$  for  $j = 1, 2, 3, 4$ . Then all distinct self-dual negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 30 are given by:

$$\mathcal{C} = \sum_{j=1}^4 \varepsilon_j(x) C_j \pmod{x^{30} + 1},$$

where  $C_j$  is an ideal of the ring  $\mathcal{K}_j + u\mathcal{K}_j$  such that:

(i) the pair  $(C_1, C_3)$  of ideals is given by one of the following five cases.

(i-1)  $5^2 = 25$  pairs:

$C_1 = \langle (x + 2)b(x) + u \rangle$  and  $C_3 = \langle 3x^{29}(x + 3)b(x^{-1}) + u \rangle$ , where

$$b(x) \in (x+2)^2(\mathcal{K}_1/\langle(x+2)^4\rangle) = \{g \cdot (x+2)^2 + h \cdot (x+2)^3 \mid g, h \in \mathbb{F}_5\}.$$

(i-2)  $5^2 + 5^1 + 5^1 + 5^0 = 36$  pairs:

$C_1 = \langle(x+2)^{k+1}b(x) + u(x+2)^k\rangle$  and  $C_3 = \langle 3x^{29}(x+3)b(x^{-1}) + u, (x+3)^{5-k}\rangle$ , where  $b(x) \in (x+2)^{\lceil \frac{1}{2}(5-k-2) \rceil}(\mathcal{K}_1/\langle(x+2)^{5-k-1}\rangle)$  and  $1 \leq k \leq 4$ .

(i-3) 6 pairs:  $C_1 = \langle(x+2)^k\rangle$  and  $C_3 = \langle(x+3)^{5-k}\rangle$ , where  $0 \leq k \leq 5$ .

(i-4)  $5^0 + 5^1 + 5^1 + 5^2 = 36$  pairs:

$C_1 = \langle(x+2)b(x) + u, (x+2)^t\rangle$  and  $C_3 = \langle 3x^{29}(x+3)^{5-t+1}b(x^{-1}) + u(x+3)^{5-t}\rangle$ , where  $b(x) \in (x+2)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_1/\langle(x+2)^{t-1}\rangle)$ ,  $1 \leq t \leq 4$ .

(i-5)  $(5^0 + 5^1 + 5^1) + (5^0 + 5^1) + 5^0 = 18$  pairs:

$C_1 = \langle(x+2)^{k+1}b(x) + u(x+2)^k, (x+2)^{k+t}\rangle$  and

$$C_3 = \langle 3x^{29}(x+3)^{5-k-t+1}b(x^{-1}) + u(x+3)^{5-k-t}, (x+3)^{5-k}\rangle,$$

where  $b(x) \in (x+2)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_1/\langle(x+2)^{t-1}\rangle)$ ,  $1 \leq t \leq 5-k-1$  and  $1 \leq k \leq 3$ .

(ii)  $(C_2, C_4)$  is given by one of the following five cases.

(ii-1)  $5^{2 \cdot 2} = 625$  pairs:

$C_2 = \langle(x^2 + 2x + 4)b(x) + u\rangle$  and  $C_4 = \langle 3x^{28}(x^2 + 3x + 4)b(x^{-1}) + u\rangle$ , where  $b(x) \in (x^2 + 2x + 4)^2(\mathcal{K}_2/\langle(x^2 + 2x + 4)^4\rangle) = \{(g_0 + g_1x) \cdot (x^2 + 2x + 4)^2 + (h_0 + h_1x) \cdot (x^2 + 2x + 4)^3 \mid g_0, g_1, h_0, h_1 \in \mathbb{F}_5\}$ .

(ii-2)  $5^{2 \cdot 2} + 5^{2 \cdot 1} + 5^{2 \cdot 1} + 5^{2 \cdot 0} = 676$  pairs:

$C_2 = \langle(x^2 + 2x + 4)^{k+1}b(x) + u(x^2 + 2x + 4)^k\rangle$  and  $C_4 = \langle 3x^{28}(x^2 + 3x + 4)b(x^{-1}) + u, (x^2 + 3x + 4)^{5-k}\rangle$ , where  $b(x) \in (x^2 + 2x + 4)^{\lceil \frac{1}{2}(5-k-2) \rceil}(\mathcal{K}_2/\langle(x^2 + 2x + 4)^{5-k-1}\rangle)$  and  $1 \leq k \leq 4$ .

(ii-3) 6 pairs:  $C_2 = \langle(x^2 + 2x + 4)^k\rangle$  and  $C_4 = \langle(x^2 + 3x + 4)^{5-k}\rangle$ , where  $0 \leq k \leq 5$ .

(ii-4)  $5^{2 \cdot 0} + 5^{2 \cdot 1} + 5^{2 \cdot 1} + 5^{2 \cdot 2} = 676$  pairs:

$C_2 = \langle(x^2 + 2x + 4)b(x) + u, (x^2 + 2x + 4)^t\rangle$  and  $C_4 = \langle 3x^{28}(x^2 + 3x + 4)^{5-t+1}b(x^{-1}) + u(x^2 + 3x + 4)^{5-t}\rangle$ , where  $b(x) \in (x^2 + 2x + 4)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_2/\langle(x^2 + 2x + 4)^{t-1}\rangle)$ ,  $1 \leq t \leq 4$ .

(ii-5)  $(5^{2 \cdot 0} + 5^{2 \cdot 1} + 5^{2 \cdot 1}) + (5^{2 \cdot 0} + 5^{2 \cdot 1}) + 5^{2 \cdot 0} = 78$  pairs:

$C_2 = \langle(x^2 + 2x + 4)^{k+1}b(x) + u(x^2 + 2x + 4)^k, (x^2 + 2x + 4)^{k+t}\rangle$  and  $C_4 = \langle 3x^{28}(x^2 + 3x + 4)^{5-k-t+1}b(x^{-1}) + u(x^2 + 3x + 4)^{5-k-t}, (x^2 + 3x + 4)^{5-k}\rangle$ , where  $b(x) \in (x^2 + 2x + 4)^{\lceil \frac{t-2}{2} \rceil}(\mathcal{K}_2/\langle(x^2 + 2x + 4)^{t-1}\rangle)$ ,  $1 \leq t \leq 5-k-1$  and  $1 \leq k \leq 3$ .

Therefore, the number of self-dual negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 30 is equal to  $121 \cdot 2061 = 249381$  and the number of all negacyclic codes over  $\mathbb{F}_5 + u\mathbb{F}_5$  of length 30 is equal to  $121^2 \cdot 2061^2 = 62190883161$ .

**Remark** Using conclusions in [3], one can determine all distinct negacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $2p^s l^t$  and their dual codes, for any odd prime  $l$  coprime to  $p$  and positive integer  $t$ .

For any  $\alpha, \beta \in \mathbb{F}_{p^m}^\times$  and positive integer  $n$  satisfying  $\gcd(p, n) = 1$ ,  $(\alpha + u\beta)$ -constacyclic codes and their dual codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$  of length  $p^s$  has been considered by another paper, when  $p$  is any odd prime.

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